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AN ELEMENTARY ESSAY ON THE COMPUTATION OF LOGARITHMS;

INTENDED AS

A SUPPLEMENT

TO THE

USUAL BOOKS ON ALGEBRA.

BY J. R. YOUNG.

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PREFACE.

THIS little tract is designed chiefly as a sequel to the chapter on Logarithms, in my own Treatise on Algebra, where, as is usual in books on that branch of analysis, the method of computing logarithms is but very briefly touched upon.

To students desirous of obtaining more detailed information about the construction of these very important numbers than such books supply, I am in hopes that the present Essay may prove acceptable, although I do not pretend to have condensed within the compass of these few pages *all* that might be worth knowing on this subject. I have proposed to myself merely to collect together and briefly, but clearly, to explain some of the best formulas and processes which mathematicians have devised for facilitating the construction of a table of logarithms. Some such explanation seems indeed wanting, on account of the very concise and inadequate directions generally given, in elementary books, on the subject of *computing* logarithms; and I shall feel happy if the present little Essay tend in any degree to supply a deficiency in so interesting and important a department of mathematical inquiry.

It may be proper to mention here, that, besides the books referred to in the following pages, I have also consulted, with advantage, *Delambre's* preface to *Borda's Tables*, the tables of *Callet*, the *Analyse Algébrique* of *Garnier*, *Hellins's Mathematical Essays*, and *Mr. Manning's* ingenious paper in the *Philosophical Transactions* for 1806. There will, however, I think, be found, upon examination, several original remarks and improved processes, together with the correction of one or two errors of some consequence. The defect, pointed out at p. 60, in *Lagrange's* method of approximating to the greatest root of an algebraical equation has, I believe, been hitherto concealed, and thus an undue superiority has been attributed to this method over that of *Newton*. This defect is here removed, and a correct process of approximation given.

J. R. Y.

November 1830.

ON THE

COMPUTATION OF LOGARITHMS.

In our treatise on Algebra, we have pretty copiously explained the nature and use of logarithms, and by means of the fundamental series established at page 206, have shown how a table of these numbers might be commenced, and carried on to any extent. We have not, however, thought proper to dwell at any length, in that place, upon the various artifices that may be advantageously adopted to abridge the labour of such a computation; our principal object having been to exhibit to the student, in a simple form, the *principles* upon which the construction of logarithmic tables depend. We here propose to resume the subject, and to present a view of some of the best methods for facilitating the construction of a table.

There are three distinct modes of computing logarithms: first, we may commence by determining the logarithm of 2, and then derive the logarithm of 3, of 5, &c., each succeeding logarithm being found by the help of those already computed; or, secondly, we may commence at any number in the table, compute its logarithm, as also the logarithms of numbers at certain intervals from this, and then interpolate the intermediate logarithms; or, lastly, the several primes may be each computed by a direct process, independently of other logarithms. These three methods we shall now explain.

CHAPTER I.

METHODS FOR COMPUTING THE LOGARITHMS OF THE PRIME NUMBERS IN SUCCESSION.

(1.) We shall commence by establishing the fundamental logarithmic series already adverted to.

Let N be any given number, and a the base of any system of logarithms; then, in order to determine the logarithm of N, which logarithm we shall denote by x, we have the equation $a^x = \mathrm{N}$. To effect this determination, substitute $1+m$ for a, and $1+n$ for N, and our equation will then appear under the form $(1+m)_x = 1+n$, from which we immediately deduce the equation $(1+m)^{xy} = (1+n)^y$, whatever be the value of y. Hence, developing each member by the binomial theorem, we have

$$1 + xym + \frac{xy\,(xy-1)}{2}m^2 + \frac{xy\,(xy-1)\,(xy-2)}{2\cdot 3}m^3 + \&\text{c.} =$$

$$1 + yn + \frac{y\,(y-1)}{2}n^2 + \frac{y(y-1)\,(y-2)}{2\cdot 3}n^3 + \&\text{c.}$$

or, expunging the 1 from each side, and dividing by y,

$$x\,\{m + \frac{xy-1}{2}m^2 + \frac{(xy-1)\,(xy-2)}{2\cdot 3}m^3 + \&\text{c.}\} =$$

$$n + \frac{y-1}{2}n^2 + \frac{(y-1)\,(y-2)}{2\cdot 3}n^3 + \&\text{c.}$$

Now since this is true, whatever be the value of y, it is true when $y = 0$, in which case the equation becomes

$$x\,(m - \tfrac{1}{2}m^2 + \tfrac{1}{3}m^3 - \tfrac{1}{4}m^4 + \&\text{c.}) = n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \&\text{c.}$$

whence

$$x = \log.\,(1+n) = \frac{n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \&\text{c.}}{m - \frac{1}{2}m^2 + \frac{1}{3}m^3 - \frac{1}{4}m^4 + \&\text{c.}}$$

that is, replacing m by its value $a-1$,

$$\log.\,(1+n) = \frac{n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \&\text{c.}}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \&\text{c.}}$$

which is an expression for the logarithm of any number, $1+n$, in terms of that number, and of the assumed base, a. The denominator of this expression is, we see, constant, that is, it remains the same for every value of the number $1+n$, provided that the base, a, does not alter. It is the reciprocal of this denominator that we have agreed to call the *modulus* of the system* that is

$$M = \frac{1}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \&c.}$$

so that

$$\log.\ (1+n) = M\,(n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \&c.)\ .\quad .\ (1.)$$

and this is the fundamental expression which we proposed to establish, and from which we shall derive the several formulas requisite to facilitate the actual construction of a table of logarithms.

(2.) If, in this expression, we make n negative, it becomes changed into

$$\log.\ (1-n) = M\,(-n - \tfrac{1}{2}n^2 - \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 - \&c.)\ .\ .\ .\ .\ (2.)$$

and if again, in each of these, we substitute $\frac{1}{p}$ for n, there results these two formulas, viz.

$$\log.\ \frac{p+1}{p} = M\,\{\frac{1}{p} - \frac{1}{2p^2} + \frac{1}{3p^3} - \frac{1}{4p^4} + \&c.\}$$

$$\log.\ \frac{p-1}{p} = -M\{\frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \&c.\}$$

from which we immediately get

$$\log.\ (p+1) = \log.\ p + M\,\{\frac{1}{p} - \frac{1}{2p^2} + \frac{1}{3p^3} - \frac{1}{4p^4} + \&c.\}\ ...(3.)$$

$$\log.\ (p-1) = \log.\ p - M\,\{\frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \&c.\}\ ...(4.)$$

two theorems which we shall presently find useful.

If equation (2) be subtracted from (1), there will result

---

* In the Algebra, read M for $\frac{1}{M}$ throughout the article on logarithms.

$$\log.\,(1+n) - \log.\,(1-n) = \log.\,\frac{1+n}{1-n} = 2\mathrm{M}\,(n + \tfrac{1}{3}\,n^3 + \tfrac{1}{5}\,n^6 + \tfrac{1}{7}\,n^7 + \&c.) \ldots\ldots (5,)$$

in which, if $\frac{1}{2p+1}$ be substituted for n, we shall have

$$1+n = \frac{2\,(p+1)}{2p+1},\ 1-n = \frac{2p}{2p+1}\ \therefore\ \frac{1+n}{1-n} = \frac{p+1}{p}$$

and these substitutions transform equation (5) into

$$\log.\,(p+1) - \log.\,p = 2\mathrm{M}\,\left\{\frac{1}{2p+1} + \frac{1}{3\,(2p+1)^3} + \frac{1}{5\,(2p+1)^5} + \frac{1}{7\,(2p+1)^7} + \&c.\right\} \ldots\ldots (6),$$

from which we obtain this general expression for determining the logarithm of any number, $p+1$, when we know the logarithm of the preceding number, p, viz.

$$\log.\,(p+1) = \log.\,p + 2\mathrm{M}\,\left\{\frac{1}{2p+1} + \frac{1}{3\,(2p+1)^3} + \frac{1}{5\,(2p+1)^5} + \frac{1}{7\,(2p+1)^7} + \&c.\right\} \ldots\ldots (7.)$$

The series (4) will accomplish the same object as this, but, as it is much less convergent, it is, in general, proportionally less commodious, for we must compute a much greater number of terms of the former series than of the latter, to obtain the logarithm sought to any given extent of decimals; but, in a few cases, to be presently noticed, the several terms of the former series are so much more easily computed than those of the latter, as to counterbalance the disadvantage arising from want of convergency. The series just deduced we have already employed (*Treat. on Alg. p.* 208) for determining the logarithms of 2 and 5 in the *Napierian* system, which is that in which the base a is so assumed as to render M equal to 1*, and, by means of these two logarithms, we have ascertained what value the modulus M should have in the common, or Briggs's, system, where a is assumed equal to 10: *(see Treatise referred to above.)* For the computation of the three first prime numbers, 2, 3, and 5, this series, from its rapid convergency, is tolerably well adapted; but for the higher primes it is much inferior to that which we are about to investigate; and, indeed, for the three leading primes already noticed, better series may be employed, as will be seen in the next article.

---

* The method of determining the value of this base is given in a note at the end.

(3.) We have seen, in last article, that, by adopting a very simple artifice, we have been enabled to convert *divergent* series, (1) and (5), into others, (3) and (7), of rapid convergency; this artifice consisted merely in changing n into $\frac{1}{p}$, in the one case, and into $\frac{1}{2p+1}$, in the other. Now it is easy to conceive that by means of different substitutions of this kind, we may arrive at series of various degrees of convergency; and, as the computation of a logarithm generally becomes more easy as the series which we employ becomes more convergent, it is obvious that a happy substitution may be the means of considerably abridging the labour of computation. If, instead of putting $\frac{1}{2p+1}$ for n, in (5), we had substituted $\frac{2}{p^3-3p}$ we should have been conducted to a series of very commodious application, first given by *M. Borda,* and known by the name of *Borda's formula.* Let us, then, deduce this formula, by substituting as directed.

Since, in this case

$$\frac{1+n}{1-n}=\frac{1+\frac{2}{p^3-3p}}{1-\frac{2}{p^3-3p}}=\frac{p^3-3p+2}{p^3-3p-2}=\frac{(p-1)^2(p+2)}{(p+1)^2(p-2)}$$

the equation (5) becomes

$$2\log.(p-1)+\log.(p+2)-2\log.(p+1)-\log.(p-2)=$$
$$2M\left\{\frac{2}{p^3-3p}+\tfrac{1}{3}\left(\frac{2}{p^3-3p}\right)^3+\tfrac{1}{5}\left(\frac{2}{p^3-3p}\right)^5+\&c.\right\}\ \ldots.\ (8),$$

or, by transposition,

$$\log.(p+2)=\log.(p-2)+2\log.(p+1)-2\log.(p-1)+$$
$$2M\left\{\frac{2}{p^3-3p}+\tfrac{1}{3}\left(\frac{2}{p^3-3p}\right)^3+\tfrac{1}{5}\left(\frac{2}{p^3-3p}\right)^5+\&c.\right\}\ \ldots.\ (9.)$$

This is Borda's formula, and, when p is not very small, the series is obviously very convergent. We may employ this formula with advantage after the logarithms of the three first primes, 2, 3, and 5, have been determined.—For the logarithm of 2 this expression is quite inapplicable; for log. 3 it yields to the series (7), and for log. 5, that is, when $p=3$, it possesses no advantage over the series (7), but, in fact, becomes identical with it, so that for these three numbers the former series may be advantageously employed. But, as observed at the close of last article, the logarithms of the same numbers may be expressed by series still more convergent, and these we shall now proceed to deduce.

Substitute, in the equation (7), q^2 for $p+1$, and it becomes

$$\log.q^2=2\log.q=\log.(q^2-1)+2M\left\{\frac{2}{2q^2-1}+\frac{1}{3(2q^2-1)^3}+\frac{1}{5(2q^2-1)^5}+\&c.\right\}$$

or representing, for brevity, the series which multiplies 2M by S, and recollecting that $q^2-1=(q+1)(q-1)$, we have, by transposing,

$$\log.(q+1)=2\log.q-\log.(q-1)-2MS.$$

Let us now suppose q to be 4, 5, 9, successively, and let S′, S″, S‴, represent the corresponding values of S, we shall thus have these three equations, viz.

$$\log.5\ =2\log.4-\log.3-2MS'$$

$$\log.\ 6 = 2\log.\ 5 - \log.\ 4 - 2MS''$$
$$\log.\ 10 = 2\log.\ 9 - \log.\ 8 - 2MS'''$$

or, substituting 4 log. 2 for 2 log. 4, log. 3 + log. 2 for log. 6, 2 log. 2 for log. 4, log. 5 + log. 2 for log. 10, 4 log. 3 for 2 log. 9, and 3 log. 2 for log. 8, we have, after transposing

$$\begin{aligned}
\log.\ 5 - 4\log.\ 2 + \log.\ 3 &= -2MS' \\
\log.\ 3 + 3\log.\ 2 - 2\log.\ 5 &= -2MS'' \\
\log.\ 5 + 4\log.\ 2 - 4\log.\ 3 &= -2MS'''
\end{aligned}$$

From these three equations, the values of the three unknown quantities, log. 2, log. 3, and log. 5, may be readily deduced, (*see Alg. p.* 68); they will be found to be as follows:

$$\begin{aligned}
\log.\ 2 &= 2M\,(\ 7\,S' + \ 5\,S'' + 3\,S''') \\
\log.\ 3 &= 2M\,(11\,S' + \ 8\,S'' + 5\,S''') \\
\log.\ 5 &= 2M\,(16\,S' + 12\,S'' + 7\,S''')
\end{aligned}$$

and, substituting the values of the series S′, S″, S‴, these equations become

$$\begin{aligned}
\log.\ 2 = 2M\ \{\ 7\,&\left(\frac{1}{31} + \frac{1}{3\cdot 31^3} + \frac{1}{5\cdot 31^5} + \frac{1}{7\cdot 31^7} + \&c.\right) \\
+\ 5\,&\left(\frac{1}{49} + \frac{1}{3\cdot 49^3} + \frac{1}{5\cdot 49^5} + \frac{1}{7\cdot 49^7} + \&c.\right) \\
+\ 3\,&\left(\frac{1}{161} + \frac{1}{3\cdot 161^3} + \frac{1}{5\cdot 161^5} + \frac{1}{7\cdot 161^7} + \&c.\right) \\
\log.\ 3 = 2M\ \{11\,&\left(\frac{1}{31} + \frac{1}{3\cdot 31^3} + \frac{1}{5\cdot 31^5} + \frac{1}{7\cdot 31^7} + \&c.\right)\} \\
+\ 8\,&\left(\frac{1}{49} + \frac{1}{3\cdot 49^3} + \frac{1}{5\cdot 49^5} + \frac{1}{7\cdot 49^7} + \&c.\right) \\
+\ 5\,&\left(\frac{1}{161} + \frac{1}{3\cdot 161^3} + \frac{1}{5\cdot 161^5} + \frac{1}{7\cdot 161^7} + \&c.\right)\} \\
\log.\ 5 = 2M\ \{16\,&\left(\frac{1}{31} + \frac{1}{3\cdot 31^3} + \frac{1}{5\cdot 31^5} + \frac{1}{7\cdot 31^7} + \&c.\right) \\
+\ 12\,&\left(\frac{1}{49} + \frac{1}{3\cdot 49^3} + \frac{1}{5\cdot 49^5} + \frac{1}{7\cdot 49^7} + \&c.\right) \\
+\ 7\,&\left(\frac{1}{161} + \frac{1}{3\cdot 161^3} + \frac{1}{5\cdot 161^5} + \frac{1}{7\cdot 161^7} + \&c.\right)\}
\end{aligned}$$

For the calculation of the single logarithm log. 2, the preceding series does not possess any great advantage, in point of practical facility, over the series (7); their superiority consists in the readiness with which they furnish the logarithms of the two succeeding primes, 3 and 5, after the logarithm of 2 has been deduced, on account of the same series recurring in each expression.

Supposing $M=1$, and adding together the Napierian logarithms of 2 and 5, as determined from these expressions, we have, for the Napierian logarithm of 10,*

$$\log.' 10 = 2 \cdot 302585092994045684,$$

and therefore

$$M = \frac{1}{\log.' 10} = \cdot 434294481903251827$$

hence, multiplying log.′ 2, log.′ 3, and log.′ 5, each by this number, we have

$$\log. 2 = \cdot 30102999566398$$
$$\log. 3 = \cdot 47712125471966$$
$$\log. 5 = \cdot 69897000433602$$

(4.) As it is necessary to know, at least, the logarithms of 2 and of 5, before Borda's theorem can be applied with advantage, we shall here make known a very neat and expeditious method of determining the logarithm of 5, when that of 2 has been computed by the series (7), or by any other independent process.† The method here mentioned is this:

Since $2^{10} = 1024 = 1000\,(1 + \frac{24}{1000})$, therefore, taking the logarithm

$$10 \log. 2 = 3 \log. 10 + \log. (1 + \tfrac{24}{1000});$$

hence, substituting $\frac{24}{1000}$ for n, in the series (1), we have, after transposing,

$$3 \log.' 10 = 10 \log.' 2 - \tfrac{24}{1000} + \tfrac{1}{2}(\tfrac{24}{1000})^2 - \tfrac{1}{3}(\tfrac{24}{1000})^3 + \&c.$$

---

* In order to distinguish Napier's logarithms from Briggs's, the former are here written with an accent.

† A direct method of calculating this logarithm will be found in the third chapter.

The calculation of this series is as follows:

$$
\begin{array}{r@{}l}
1000(24 & (\cdot\ 024 - \\
\underline{2} & \\
48 & \\
\underline{12} & \\
2 \cdot 1000^2)576 & (\cdot\ 000288 + \\
\underline{2} & \\
1152 & \\
\underline{12} & \\
3 \cdot 1000^3)13824 & (\cdot\ 000004608 - \\
\underline{2} & \\
27648 & \\
\underline{12} & \\
4 \cdot 1000^4)331776 & (\cdot\ 000000082944 + \\
\underline{2} & \\
663452 & \\
\underline{12} & \\
5 \cdot 1000^5)7962624 & (\cdot\ 000000001592552 - \\
\underline{2} & \\
15925248 & \\
\underline{12} & \\
6 \cdot 1000^6)191102976 & (\cdot\ 000000000003185 + \\
\underline{2} & \\
382205952 & \\
\underline{12} & \\
7 \cdot 1000^7)4586471424 & (\cdot\ 000000000000065 -
\end{array}
$$

Hence, collecting the negative and positive terms separately, we have

$$
\begin{array}{r@{}l}
-\ 02400460959317 & = \text{sum of negative terms.} \\
\underline{+\ 00028808297585} & = \text{sum of positive terms.} \\
-\ 02371652661732 & = \text{sum of the series.} \\
\underline{6 \cdot 93147180559945} & = 10 \text{ log.}'\ 2. \\
3)\underline{6 \cdot 90775527898213} & = 3 \text{ log.}'\ 10. \\
\underline{2 \cdot 30258509299404} & = \text{log.}'\ 10.
\end{array}
$$

We have thus got the Napierian logarithm of 10 true to fourteen places of decimals; the difference between log.′ 10 and log.′ 2, previously found, will give log.′ 5, and the modulus being now determined, we may employ the series (4) for computing log. 3. For, putting 10 for *p*, this series gives

$$\text{log. } 9 = 2 \text{ log. } 3 = \text{log. } 10 - \text{M } \{\frac{1}{10} + \frac{1}{200} + \frac{1}{3000} + \frac{1}{40000} + \&c.\}$$

whence

$$\text{log. } 3 = \tfrac{1}{2} - \tfrac{1}{2} \text{ M } \{\frac{1}{10} + \frac{1}{200} + \frac{1}{3000} + \frac{1}{40000} + \&c.\}$$

The summation of this series is very easy; we have only to divide $\frac{1}{10}$ of $\frac{1}{2}$ M, that is, the number

·021714724095161,

successively by 2, 3, 4, &c. up to 12, prefixing to each quotient as many ciphers, minus one, as there are units in the divisor, and to add them and the dividend together thus

```
·021714724095161
   1085736204758
    072382413651
     05428681024
      0434294482
       036191207
        03102103
         0271434
          024127
           02171
            0197
             018
────────────────
·022878745280333
```

This result taken from ·5 leaves for log. 3

log. 3 = ·477121254719667

which is true, as far as fourteen places of decimals.

If, in the preceding example, instead of putting 10 for *n*, we had put either of the numbers 100, 1000, 10000,

&c. we should obviously have obtained the very same series of quotients, as far as the significant figures are concerned, but each would have been preceded by a greater number of ciphers; having, then, these quotients already prepared, we may at once determine the logarithms of 99, of 999, of 9999, &c.; and, by taking the quotients alternately positive and negative, agreeably to the series (3) we may as easily compute the logarithms of 11, 101, 1001, 10001, &c.; we shall therefore consider, in what follows, that these logarithms have been determined in this place.

(5.) As we have now given a choice of methods for computing the logarithms of the three first prime numbers, we may proceed to apply Borda's theorem to the determination of the others in succession. But it is proper that we should first fix a limit to the number of decimals each logarithm is to have, as also to the extent of the table to be constructed. For all the usual purposes of calculation, a table of 100000 logarithms, of seven decimals each, is found to be amply sufficient, and accordingly to this extent all our best modern tables are carried. Previously, however, to tabulating the logarithms, it is necessary that they be carried to a greater extent of decimals than this, in order to provide against the errors which we might expect would affect the latter decimals of many logarithms determined by the combination of others, which are themselves not strictly accurate, (except when their numbers are powers of 10,) because of the omission of decimals to the right. We shall, therefore, in this chapter, suppose the logarithms to be computed in the first instance to twelve decimal places.

(6.) If we now return to Borda's theorem, and examine the second term of the series, we easily perceive that, when

$p = 39$, this term can have no significant figure in the first thirteen places of decimals, so that for the prime number 41, and all the succeeding, we need take account only of the first term of the series, and we shall now show how, by means of a few obvious transformations, we may dispense with the second term of the series in computing the primes below 41. In the following table, we suppose of course that the logarithms arising from the combination of those of 2, 3, and 5, are known, as also log. 11, (see art. 4;)

1. Let $p = 34 \therefore \log. 36 = \log. 32 + 2 \log. 35 - 2 \log. 33 + \frac{2M}{19601} + \&c.$

or $2 \log. 2 + 2 \log. 3 = 5 \log. 2 + 2 \log. 5 + 2 \log. 7 - 2 \log. 3 - 2 \log. 11 + \frac{2M}{19601} + \&c.$

but $5 \log. 2 + 2 \log. 5 = 3 \log. 2 + 2 \{\log. 2 + \log. 5\} = 3 \log. 2 + 2$, therefore

$$\log. 7 = -1 - \tfrac{1}{2} \log. 2 + 2 \log. 3 + \log. 11 - \frac{M}{19601} - \&c.$$

2. Let $p = 37 \therefore \log. 39 = \log. 13 + \log. 3 = \log. 35 + 2 \log. 38 - 2 \log. 36 + \frac{2M}{25271} + \&c.$

$\therefore \log. 13 = \log. 35 + 2 \log 19^{*} - 2 \log. 18 - \log. 3 + \frac{2M}{25271} + \&c.$

3. Let $p = 134 \therefore \log. 136 = \log. 17 + \log. 8 = 132 + 2 \log. 135 - 2 \log. 133 + \frac{2M}{1202851} + \&c.$

$\therefore \log. 17 = \log. 132 + 2 \log. 135 - 2 \log. 133 - \log. 8 + \frac{2M}{1202851} + \&c.$

4. Let $p = 97 \therefore \log. 99 = \log. 95 + 2 \log. 98 - 2 \log. 96 + \frac{2M}{456191} + \&c.$

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[*] Log. 19 will be determined independently of log. 13.

$$\therefore \log. 19 = \log. 99 - \log. 5 - 2 \log. 98 + 2 \log. 96 - \frac{2M}{456191} - \&c.$$

5. Let $p = 67 \therefore \log. 69 = \log. 23 + \log. 3 = \log. 65 + 2 \log. 68 -$

$$2 \log. 66 + \frac{2M}{150281} + \&c.$$

$$\therefore \log. 23 = \log. 65 + 2 \log. 68 - 2 \log. 66 - \log. 3 + \frac{2M}{150281} + \&c.$$

6. Let $p = 56 \therefore \log. 58 = \log. 29 + \log. 2 = \log. 54 + 2 \log. 57 -$

$$2 \log. 55 + \frac{2M}{87724} + \&c.$$

$$\therefore \log. 29 = \log. 27 + 2 \log. 57 - 2 \log. 55 + \frac{M}{43862} + \&c.$$

7. Let $p = 122 \therefore \log. 124 = \log. 31 + \log. 4 = \log. 120 + 2 \log. 123$

$$- 2 \log. 121 + \frac{2M}{907741}$$

$$\therefore \log. 31 = \log. 30 + 2 \log. 123\text{[*]} - 2 \log. 121 + \frac{2M}{907741} + \&c.$$

8. $\log. 37 = \log. \frac{999}{27} = \log. 999 - \log. 27$. *(See art. 4).*

The foregoing expressions give the several logarithms true to at least thirteen decimal places, employing only the first term of each series.

We may now continue the computation without performing any transformation on Borda's series, and without using more than the first term, so that the formula becomes simply

$$\log. (p + 2) = \log. (p - 2) + 2 \log. (p + 1) - 2 \log. (p - 1) + \frac{4M}{p^3 - 3p} \ldots\ldots (10).$$

With this formula we may compute the next thirteen prime numbers, that is, till we come to the prime 101, when it may be changed for one rather more commodious, which we deduce as follows.

(7.) In the equation (5) substitute $\frac{1}{2p^2 + 4p + 1}$ for n, and it becomes

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[*] This log. depends on log. 41, which is at once obtainable from Borda's theorem, without performing any preliminary transformation.

$$\log.\frac{p^2+2p+1}{p\,(p+2)}=2\mathrm{M}\left\{\frac{1}{2p^2+4p+1}+\frac{1}{3(2p^2+4p+1)^3}+\&\mathrm{c.}\right\}$$

but

$$p^2+2p+1=(p+1)^2,\ \text{also}\ \frac{1}{2p^2+4p+1}=\frac{\frac{1}{2}}{(p+1)^2-\frac{1}{2}};$$

hence

$$2\log.(p+1)-\log.p-\log.(p+2)=\frac{\mathrm{M}}{(p+1)^2-\cdot 5}+\&\mathrm{c.}$$

omitting the second term of the series, because, when $p=100$, this term can have no significant figure in the first thirteen places of decimals; therefore, by transposing, we have

$$\log.(p+1)=\tfrac{1}{2}\{\log.p+\log.(p+2)\}+\frac{\frac{1}{2}\mathrm{M}}{(p+1)^2-\cdot 5}\ .\ (11.)$$

This theorem is of easy application, and is, at the same time, very correct, for $p+1$ being a prime number, p and $p+2$ must be composite numbers, and, therefore, the logarithms of these latter will be determined from those nearer to the commencement of the table, than the number $p+1$, by at least one half the distance of $p+1$ from the commencement, and, of course, the nearer to the commencement the logarithms are taken, the less is the probability of errors in their last decimals. The principal labour attending the application of this formula consists in squaring the proposed number, and subsequently dividing ½M or ·2171472409516 by the result; but when the square of one number is ascertained, that of any other number, q, units higher, is found by multiplying the former by twice q, and adding q^2 to the result; this part of the work is, therefore, very trifling, and the labour of the division continually lessens, because fewer quotient figures are required as the computation proceeds. When the succeeding primes below 500 have been computed by this formula, we may substitute for it another unconnected with any series, and which is the result of the following investigation.

(8.) In the series,

$$\log.(1+n) = \text{M}\{n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \tfrac{1}{5}n^5 - \&\text{c.}\},$$

put $\frac{q}{p}$ for n, and we have

$$\log.(p+q) = \log.p + \text{M}\{\frac{q}{p} - \frac{q^2}{2p^2} + \frac{q^3}{3p^3} - \frac{q^4}{4p^4} + \frac{q^5}{5p^5} - \&\text{c.}\}$$

Let q be successively 5, 4, 3, 2, 1, and we shall then easily deduce this equation, viz.

$$\log.(p+5) - 5\log.(p+4) + 10\log.(p+3) - 10\log.(p+2) + 5\log.(p+1) =$$

$$\begin{array}{l}
\log.p + \text{M}\{\frac{5}{p} - \frac{5^2}{2p^2} + \frac{5^3}{3p^3} - \frac{5^4}{4p^4} + \frac{5^5}{5p^5} - \frac{5^6}{6p^6} + \&\text{c.}\} \\
- 5\log.p - 5\text{M}\{\frac{4}{p} - \frac{4^2}{2p^2} + \frac{4^3}{3p^3} - \frac{4^4}{4p^4} + \frac{4^5}{5p^5} - \frac{4^6}{6p^6} + \&\text{c.}\} \\
+ 10\log.p + 10\text{M}\{\frac{3}{p} - \frac{3^2}{2p^2} + \frac{3^3}{3p^3} - \frac{3^4}{4p^4} + \frac{3^5}{5p^5} - \frac{3^6}{6p^6} + \&\text{c.}\} \\
- 10\log.p - 10\text{M}\{\frac{2}{p} - \frac{2^2}{2p^2} + \frac{2^3}{3p^3} - \frac{2^4}{4p^4} + \frac{2^5}{5p^5} - \frac{2^6}{6p^6} + \&\text{c.}\} \\
+ 5\log.p + 5\text{M}\{\frac{1}{p} - \frac{1^2}{2p^2} + \frac{1^3}{3p^3} - \frac{1^4}{4p^4} + \frac{1^5}{5p^5} - \frac{1^6}{6p^6} + \&\text{c.}\} \\
\hline
= \log.p \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; + \frac{24\text{M}}{p^5} - \frac{300\text{M}}{p^6} + \&\text{c.}
\end{array}$$

consequently

$$\log.(p+5) = 5\log.(p+4) - 10\log.(p+3) + 10\log.(p+2) - 5\log.(p+1) + \log.p + \frac{24\text{M}}{p^5} - \frac{300\text{M}}{p^6} + \frac{2400\text{M}}{p^7} - \&\text{c.} \;.\;.\;.\;. \quad (12.)$$

Now, when $p = 500$, the first term of this series is too small to affect the twelfth decimal, so that, for primes above 500, the series may be entirely rejected, without causing an error sufficiently great to affect the accuracy of the twelve first decimals in the resulting logarithm. Hence the formula for computation is

$$\tfrac{1}{3}\left(\frac{72}{x^4-25\,x^2+72}\right)^3 + \&c.\}$$

The convergency of this series, when x is not a small number, exceeds the convergency of Borda's series, but for primes below 53, Borda's converges fastest. This formula of Haros will give the logarithm of 53 true to thirty decimals, by employing only the two leading terms of the series, and when $x = 1000$, supposing the preceding logarithms to have been calculated, we shall get the logarithm of the prime number 1005 to thirty places, by using only the first term.

(11.) Before closing the present chapter, it may be proper to apprise the student that some caution is necessary in deducing logarithmic formulas one from another, in order to avoid erroneous conclusions. Thus, although the expression $\frac{1+n}{1-n}$ involves no impossibility, whatever number we put for n, whether positive or negative, it is not so when we take the logarithm of the same expression, for log. $\frac{1+n}{1-n}$ is impossible for all positive values of n greater than 1, and for all negative values less than 1; in other words, this expression can have a real signification only when n is either a positive or negative proper fraction. This obviously results from the fact that no negative number can have a real logarithm in any system whose base is a positive number. *(See chap.* iv. *following.)*

From not attending to this circumstance, some writers have fallen into error, in attempting to deduce the series of Borda and Haros directly from the series (5),* for they substitute, in this formula, $p^3 - 3p + 2$ for $1 + n$, and $p^3 - 3p - 2$ for $1 - n$, in order to get Borda's formula, and to deduce that of Haros, the expressions $x^4 - 25x^2 + 144$, and $x^4 - 25x^2$, are substituted. The series resulting from these substitutions are indeed the same in form as those of Borda and Haros, but they are not the series themselves for they do not even converge, but, on the contrary, diverge.

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* Mr. Jephson, at p. 54, vol. 1, of his Fluxional Calculus, has committed this inadvertency. We are happy to be furnished with even this opportunity of referring to Mr. Jephson's work, the second volume of

CHAPTER II.

ON THE COMPUTATION OF LOGARITHMS BY THE METHOD OF DIFFERENCES, OR BY INTERPOLATION.

(12.) The method of differences, as far as regards the computation of logarithms, may be explained as follows: Let a, b, c, d, e, &c. represent any series of numbers; then, if their successive differences be taken, these differences will form a new series, viz. $a - b$, $b - c$, $c - d$, $d - e$, or else $b - a$, $c - b$, $d - e$, $e - d$, accordingly as the numbers in the proposed series decrease or increase. This new series is called the *first order of differences.* In like manner, if the successive differences of the terms of this last series be taken, a new series, called the *second order of differences,* will be obtained, and so on. Here we must remark, that although the original series may be a decreasing one, yet the successive orders of differences

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which is a valuable addition to our scanty stock of books on the Integral Calculus, and well deserves the attention of students in the higher departments of analysis.

may all be increasing series, and that, on the other hand, when the original series is an increasing one, the several orders of differences may all be decreasing series. This, indeed, is precisely the case with logarithms; taken in an increasing order, their successive differences diminish, and, therefore, when taken in the contrary order, their differences increase.

Let us suppose a, b, c, d, e, &c. to be logarithms taken in a decreasing order, then, for the several differences, we have

$$\begin{array}{llllll}
 & a,\ \ b, & c, & d, & e, & \&c. \\
\text{1st order of diff.} & a-b, & b-c, & c-d, & d-e, & \&c. \\
 & & \underline{a-b} & \underline{b-c} & \underline{c-d} & \\
\text{2d order.} & & -a+2b-c, & -b+2c-d, & -c+2d-e, & \&c. \\
 & & & \underline{-a+2b-c} & \underline{-b+2c-d} & \\
\text{3d order.} & & & a-3b+3c-d, & b-3c+3d-e, & \&c. \\
 & & & & \underline{a-3b+3c-d} & \\
\text{4th order.} & & & & -a+4b-6c+4d-e, & \&c. \\
 & & & & & \&c.
\end{array}$$

From the mode of generating these differences, it is obvious, first, that the second difference in any order is equal to the first in the same order, *plus* the first in the succeeding order; and, secondly, that the coefficients of the several terms composing either of the differences, in any order, are respectively the same as the coefficients in that power of the expanded binomial.

Let, now, the same series be considered as increasing, then

$$\begin{array}{llllll}
 & a,\ \ b, & c, & d, & e, & \&c. \\
\text{1st ord. of diff.} & b-a, & c-b, & d-c, & e-d & \\
 & & \underline{b-a,} & \underline{c-b,} & \underline{d-c} & \\
\text{2d order.} & & -c+2b-a, & -d+2c-b, & -e+2d-c, & \&c. \\
 & & & \underline{-c+2b-a} & \underline{-d+2c-b} &
\end{array}$$

$$\begin{array}{lll} \text{3d order.} & d-3c+3b-a, & e-3d+3c-b,\ \&c. \\ & & \underline{d-3c+3b-a,} \\ \text{4th order.} & & -e+4d-6e+4b-a,\ \&c. \\ & & \&c. \end{array}$$

Here we see that the second difference in any order is equal to the first in the same order, *minus* the first in the succeeding order, and, as regards the coefficients, the same may be observed as in the preceding case.

Let us now suppose that, by continuing the above process of differencing, we at length arrive at an order of differences, of which each term differs so little from the consecutive term, that they may for a certain number of terms be considered as equal, without material error. Admit such an order of differences to be the fourth; then, when the original series is decreasing, if we add this constant fourth difference to the first of the third order, the first of the third order to the first of the second, the first of the second to the first of the first, and note the three results, and then, returning to the same constant fourth difference, add it to the first result, this first result to the second, and the second to the third, noting each new result, and then proceed with the constant fourth difference, and these results as with the former set, we shall obviously obtain, in succession, the several first differences, $b-c$, $c-d$, $d-e$, &c., and a, and $a-b$, the first term and first difference being previously known we shall obtain, by subtracting this difference from the first term, the next difference from the result, the next from this result, and so on, all the terms, b, c, d, e, &c., in succession, till we have got the same number as those for which we have supposed the fourth differences constant.* When the original series is increasing, the first differences, $c - b$, $d - c$, $e - d$, are obtained by successively subtracting where in the above case we have added, and the terms themselves are got from the first differences by adding, instead of subtracting, as before. It follows, therefore, that when the first term is given, and also the first of each order of differences, till we come to a constant difference, all the succeeding terms of the series may be found, and it is upon this that the computation of logarithms by interpolation depends.

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* If the terms of any order of differences were accurately equal, it is plain that the terms of the original series might be found to any extent; but this is not the case with the logarithmic differences.

(13.) But before we can employ this method, we must obviously possess some means of determining the first term of each order of differences from the first term of the original series, and these means we shall now proceed to deduce.

We shall here, as in last article, suppose the fourth differences equal, and shall afterwards show that the error we commit in extending this supposition to one hundred consecutive differences, is too small to produce any sensible effect upon the hundred resulting logarithms, within those limits of the table, between which we propose to apply the present method of computation.

Let $n+2$, $n+1$, n, $n-1$, $n-2$, be five consecutive numbers; then, for the several first differences of their logarithms, we shall have, *(see art.* 8,*)*

First Differences.

$$\log.(n+2) - \log.(n+1) = \mathrm{M}\{\frac{1}{n} - \frac{3}{2n^2} + \frac{7}{3n^3} - \frac{15}{4n^4} + \frac{31}{5n^5} - \frac{63}{6n^6} + \&c.\}$$

$$\log.(n+1) - \log.n = \mathrm{M}\{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \frac{1}{5n^5} - \frac{1}{6n^6} + \&c.\}$$

$$\text{log. } n - \text{log. } (n-1) = \text{M}\,\{\frac{1}{n}+\frac{1}{2n^2}+\frac{1}{3n^3}+\frac{1}{4n^4}+\frac{1}{5n^5}+\frac{1}{6n^6}+\text{\&c.}\}$$

$$\text{log. } (n-1) - \text{log. } (n-2) = \text{M}\,\{\frac{1}{n}+\frac{3}{2n^2}+\frac{7}{3n^3}+\frac{15}{4n^4}+\frac{31}{5n^5}+\frac{63}{6n^6}+\text{\&c.}\}$$

Subtracting each of these from the succeeding, we have, for

Second Differences,

$$\text{M}\,\{\frac{1}{n^2}-\frac{2}{n^3}+\frac{7}{2n^4}-\frac{6}{n^5}+\frac{31}{3n^6}+\text{\&c.}\}$$

$$\text{M}\,\{\frac{1}{n^2}+\frac{1}{2n^4}+\frac{1}{3n^6}+\text{\&c.}\}$$

$$\text{M}\,\{\frac{1}{n^2}+\frac{2}{n^3}+\frac{7}{2n^4}+\frac{6}{n^5}+\frac{31}{3n^4}+\text{\&c.}\}$$

Subtracting each of these from the succeeding, we have, for

Third Differences.

$$\text{M}\,\{\frac{2}{n^3}-\frac{3}{n^4}+\frac{6}{n^5}-\frac{10}{n^6}+\text{\&c.}\}$$

$$\text{M}\,\{\frac{2}{n^3}+\frac{3}{n^4}+\frac{6}{n^5}+\frac{10}{n^6}+\text{\&c.}\}$$

and subtracting the first of these from the second, we have, for the

Fourth Difference,

$$\text{M}\,\{\frac{6}{n^4}+\frac{20}{n^6}+\text{\&c.}\}$$

As this last is to be considered constant for several consecutive terms in the fourth order, we should have arrived at an expression of the same numerical value, if we had differenced the series $n+1$, n, $n-1$, $n-2$, $n-3$, that is, if we had supposed the series to begin at $n+1$ instead of at $n+2$; we shall make this supposition, then, because the expressions for the first, second, and third differences will be those in the above table, which are most easy to calculate. Hence the logarithms of the numbers $n+1$, n, $n-1$, &c. will correspond to the letters a, b, c, &c. employed in art. (12), and the first term of the several orders of differences, which we shall represent by Δ^1, Δ^2, Δ^3, Δ^4, successively, will, by the foregoing table, be

$$\Delta^1 = \mathrm{M}\,\{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \frac{1}{5n^5} - \frac{1}{6n^6} + \&c.\} = a - b$$

$$\Delta^2 = \mathrm{M}\,\{\frac{1}{n^2} + \frac{1}{2n^4} + \frac{1}{3n^6} + \&c.\} = -a + 2b - c$$

$$\Delta^3 = \mathrm{M}\,\{\frac{2}{n^3} - \frac{3}{n^4} + \frac{6}{n^5} - \frac{10}{n^6} + \&c.\} = a - 3b + 3c - d$$

$$\Delta^4 = \mathrm{M}\,\{\frac{6}{n^4} + \frac{20}{n^6} + \&c.) = -a + 4b - 6c + 4d - e$$

From inspecting this table of differences, it appears that, when the first difference Δ^1 is calculated, the others are readily deduced. Thus

$\Delta^2 =$ Double the sum of the negative terms of Δ^1

$\Delta^3 = 6$ (3d term of Δ^1) $+ 12$ (4th term) $+ 30$ (5th term) $+ 60$ (6th term).

$\Delta^4 =$ double the sum of the negative terms of Δ^3.

Having in this way got all the first differences, we deduce, by successive additions, as explained in art. (12), the succeeding differences, $b - c$, $c - d$, &c. of the first order, and thence, by successive subtractions, the terms b, c, d, &c., a, or log. $(n+1)$, being the known logarithm from which we set out. We may now easily estimate the extent of error introduced, by supposing the fourth differences constant, within any proposed limits of the table; thus, taking $(n+1) = 100000$, which we shall consider as the last number in the table, we have for Δ^4 the value

$$\cdot 000000000000000000026,$$

and when $n + 1 = 50000$, the value is

$$\cdot 000000000000000000417;$$

we thus see by how very small a quantity the differences of the fourth order must vary from each other within these limits of the table, for the sum of all these variations, when carried through 50000 terms or logarithms, amount only to the difference of the two values of Δ^4 just determined, that is, to the number

$$\cdot 000000000000000000391,$$

a quantity in itself too small to deserve notice. It must, however, be remembered, that although each of the differences of the fourth order vary from the next on the average only by about the 50000th part of this very small number, yet it would not be safe to suppose this variation 0 through so many as 50000 terms, for the very small error affecting Δ^4 would, by continual addition, multiply itself so much, as at length to become of consequence, and, indeed, were Δ^4 affected with no such error, the same objection would hold against calculating so many terms, because as Δ^4 is not a terminating decimal, we could never employ it unaffected with any error, since we should be compelled to omit decimals to the right, which would, in like manner, at length cause sensible error in the resulting logarithms. We shall now, therefore, endeavour to ascertain how many logarithms below 100000 we may safely compute, without being obliged to take Δ^4 with an unmanageable number of decimals.

(14.) Referring to the expressions for Δ^1, Δ^2, Δ^3, &c. art. (12), we have, by transposition,

$$\begin{array}{llll}
b = & a - \Delta', \therefore, \text{by substitution}, \; . \; . \; . \; . & b = a - \Delta^1 & \\
c = & -a + 2b - \Delta^2 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . & c = a - 2\Delta^1 - \Delta^2 & \\
d = & a - 3b + 3c - \Delta^3 \; . \; . \; . \; . \; . \; . \; . & d = a - 3\Delta^1 - \Delta^2 - \Delta^3 & \\
e = & -a + 4b - 6c + 4d - \Delta^4 \; . \; . \; . \; . \; . & e = a - 4\Delta^1 - 6\Delta^2 - 4\Delta^3 - \Delta^4 & \\
f = & a - 5b + 10c - 10d + 5e + \Delta^5 \; . \; . & f = a - 5\Delta^1 - 10\Delta^2 - 10\Delta^3 - 5\Delta^4 - \Delta^5 & \\
\&c. & \&c. & \&c. & \&c.
\end{array}$$

and generally for any term l removed m places from a, the expression is obviously

$$l = a - m\Delta^1 - \frac{m\,(m-1)}{2}\Delta^2 - \frac{m\,(m-1)\,(m-2)}{2 \cdot 3}\Delta^3 - \frac{m\,(m-1)\,(m-2)\,(m-3)}{2 \cdot 3 \cdot 4}\Delta^4.$$

the terms following Δ^4 vanishing, because Δ^4 is considered constant. If we had taken the second series of expressions for Δ^1, Δ^2, Δ^3, &c. in art. (12), the proposed series being then increasing, the general expression would have been

$$l = a + m\Delta^1 - \frac{m(m-1)}{2}\Delta^2 + \frac{m(m-1)(m-2)}{2\cdot 3}\Delta^3 - \frac{m(m-1)(m-2)(m-3)}{2\cdot 3\cdot 4}\Delta^4.$$

The coefficients of Δ^1, Δ^2, Δ^3, and Δ^4, denote the number of times these differences are repeated in going from a to l, m units distant; hence, making m successively equal to 10, 20 30, &c., we have the following table of values:

val. of m	coef. of Δ^2	coef. of Δ^3	coef. of Δ^4
10	45	120	210
20	190	1140	4845
30	435	4060	27405
40	780	9880	91390
50	1225	19600	230300
&c.	&c.	&c.	&c.

We see from this table that to compute 50 logarithms on either side of the given logarithm, Δ^4, must be repeated, either by way of addition or subtraction 230300 times, so that, in the 50th logarithm, the original error in Δ^4 must become so many times increased; but that part of this error arising from our having taken Δ^4 for the constant differences of the fourth order, is, even when thus increased, still too small to affect the accuracy of the 50th logarithm, computed to 17 places of decimals. We have, therefore, only to take care to calculate Δ^4 to six places of decimals beyond that where we propose our logarithms to terminate, for then 230300 times, the error in Δ^4, will produce no effect upon this terminating decimal, provided it occupy any place below the eighteenth.*

(15.) Having now explained the principles of the method of interpolation, we may proceed at once to the actual construction of a table; first, however, calculating by the precepts in (13), the values of Δ^1, Δ^2, Δ^3, Δ^4, when $n + 1 = 100000$. These values we find to be

$$\begin{aligned}\Delta^1 &= \cdot 0000043429665339\\ \Delta^2 &= \cdot 00000000004343031679\\ \Delta^3 &= \cdot 000000000000000868628\\ \Delta^4 &= \cdot 0000000000000000000261\end{aligned}$$

and the following is a specimen of the table, in which all the logarithms are true to 16 decimals. The digits to the right of the comma in Δ^3 and Δ^2 fall beyond the last digit in the preceding column, and are, therefore, to be rejected in the addition.

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* It may be remarked that the two errors with which Δ^4 is affected oppose each other.

Numbers.	Logarithms.	Δ^1	Δ^2	Δ^3	Δ^4
100000	5 · 0000000000000000000	4342966533902	43430316,79	086862,8	0000026
99999	4 · 9999956570334666098	4343009964219	43431185,42	086865,4	0000026
99998	4 · 9999913140235018879	4343053395404	43432054,07	086868,0	0000026
99997	4 · 9999869709701006475	4343096827458	43432922,75	086870,6	0000026
99996	4 · 9999826278732790017	4343140260381	43433791,46	086873,2	0000026
99995	4 · 9999782847330186636	4343183694172	43434660,19	086875,8	0000026
99994	4 · 9999739415493244464	4343227128832	43435528,95	086878,4	0000026
99993	4 · 9999695983221956632	4343270564361	43436397,73	086881,0	&c.
99992	4 · 9999652550516631271	4343314000759	43437266,54	&c.	
99991	4 · 9999609117376330512	4343357438026	&c.		
99990	4 · 9999565683809124866	&c.			
&c. down	&c.				
to					
99949					

Had we wished our table to extend beyond the number 100000, we could just as readily have obtained 50* more logarithms by calculating upwards, the differences being derived by continual subtraction, and the logarithms found by addition. The first step of this upward work is here exhibited

100050					
&c. up to	&c.				
		&c.			
100001	5 · 000004342923104454		&c.		
		4342923104454		&c.	
100000	5 · 000000000000000000		43429448,19		&c.
		4342966533902		086860,2	
			43430316,79		0000026
				086862,8	
					0000026

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* If we employ the same given first differences in the upward calculation as in the downward, we ought to obtain fewer logarithms by

(16.) To continue the table downwards, it will be necessary to begin afresh, setting out with the number 99900, the logarithm of which we shall shortly show how to find, and calculating a new set of differences. This done, we may compute, by working both upwards and downwards, as in the preceding specimens, 100 more logarithms. In a similar manner, by setting out from the logarithm of 99800, previously calculated, we shall deduce a hundred more logarithms, and so on, down to the number 50000, that is, through half the extent of the table. By means of log. 100000 and log. 50000, we may obtain log. 2, and, taking log. 2 from all the logarithms in succession, now calculated, the remainders will obviously be the logarithms of all the numbers from 50000 to 25000. In like manner, the log. 2 taken from each of these will furnish the logarithms to the numbers between 25000 and 125000, and lastly, by taking log. 2 from those between 25000 and 20000, we obtain the logarithms to all the numbers down to 10000. As to the logarithms of the numbers below this down to unity, they will, it is plain, differ from every tenth logarithm from the top of the table only in the index, they are, therefore, all calculated, and in this manner is the table completed. In carrying on this work of interpolation, it is necessary, as we have seen above, to be provided with the logarithms of the numbers 99900, 99800, and so on at intervals of 100, and we shall now show how to compute them.

Referring to the formulas at (13), let Δ^1, Δ^2, and Δ^3, be calculated by them for $n + 1 = 1000$, and let also, seven values of Δ^4 be calculated for the following successive

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three, for the numbers should then be considered as commencing at 99997; but an additional log. or two may be safely deduced.

values of n, viz. $n = 998$,* $n = 997$, &c., the first of these will complete the first set of differences, and thence, by means of the six following values of Δ^4, we may readily find in succession the logarithms of the numbers 999, 998, 997, &c. to 990, as follows:

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* The formula for the fourth difference in art. (13) arose from differencing the series $n + 2$, $n + 1$, n, $n - 1$, $n - 2$; hence, since the differences to be calculated above are not supposed to be constant, $n + 2$ must equal the successive numbers 1000, 999, &c. from which we set out.

Numbers.	Logarithms.	Δ^1	Δ^2	Δ^3	Δ^4
1000	3 · 0000000000000000	434511774017690	435164593508	872510682,2	26267264,7
999	2 · 9995654882259823 1	434946938611198	436037104190	875137408,7	26372808,7
998	2 · 9991305412873711 1	435382975715388	436912241599	877774689,6	26478883,4
997	2 · 9986951583116557 2	435819887956987	437790016289	880422577,9	26585491,9
996	2 · 9982593384236987 3	436257677973276	438670438867	883081127,1	26692637,5
995	2 · 9978230807457254 5	436696348412143	439553519994	885750390,9	26800323,5
994	2 · 9973863843973133 1	437135901932137	440439270385	888430423,3	26908553,0
993	2 · 9969492484953811 7	437576341202522	441327700808	891121278,6	
992	2 · 9965116721541786 5	438017668903330	442218822087		
991	2 · 9960736544852753 2	438459887725417			
990	2 · 9956351945975499 0				

Now, by adding 2 to the index of each of these logarithms, there will be obtained the logarithms of 99900, 99800, &c. down to 99000, and every hundredth logarithm below this may be derived as fast as they are wanted, by the help of those previously computed. For, since

$$990 \times 101 = 99990,$$

and since, also, the logarithms of 990 and 99990 have been determined, log. 101 is known; moreover

$$\left.\begin{array}{r} 989 \\ 988 \\ 987 \\ 986 \\ 985 \\ 984 \\ 983 \\ 982 \\ 981 \\ 980 \end{array}\right\} \times 101 = \left\{\begin{array}{l} 99889 \\ 99788 \\ 99687 \\ 99586 \\ 99485 \\ 99384 \\ 99283 \\ 99182 \\ 99081 \\ 98980 \end{array}\right.$$

and log. 102 is given, because 980 × 102 = 99960, and log. 9960 is known, therefore

$$\left.\begin{array}{r} 979 \\ 978 \\ 977 \\ 976 \\ 975 \\ 974 \\ 973 \\ 972 \\ 971 \\ 970 \end{array}\right\} \times 102 = \left\{\begin{array}{l} 99858 \\ 99756 \\ 99654 \\ 99552 \\ 99450 \\ 99348 \\ 99246 \\ 99144 \\ 99042 \\ 98940 \end{array}\right.$$

from this last we get log. 103, because 970 × 103 = 99910, and so on. The logarithms of all numbers from 990 down to 500 being obtained in this easy manner, we shall have merely to add 2 to the index of each, in order to obtain the logarithm of every hundredth number down to 50000, and they will be true as far as sixteen places of decimals.

If the logarithms are not required to so many as fifteen or sixteen decimals, the work of interpolation will evidently be shortened; thus, if about ten or eleven decimals suffice, then we may all along reject the fourth differences, and consider the third as constant.

(17.) But perhaps the readiest way of all to compute a table would be, to combine with the methods delivered in this chapter some of the formulas and processes recommended in the last. Suppose, for instance, we wanted logarithms to ten decimals, which, as before remarked, is an extent more than sufficient for usual purposes, we might proceed with the computation according to the following directions.

1. Commencing at the number 90000, compute, by interpolation, with three orders of differences the logarithms down to 60000; then, if we diminish the index of log. 90000 by 4, we shall have log. 9,* the half of which will give log. 3. In like manner, from log. 60000 we get log. 6, and log. 6 − log. 3 = log. 2.

2. Subtract log. 2 from each of the logarithms of the even numbers thus computed, and subtract log. 3 from the first, and from every fourth logarithm following, and we shall obviously thus obtain the logarithms of all numbers from 45000 down to 20000, both inclusive; and again, subtracting log. 2 from each of the logarithms of the even numbers from 40000 to 20000, we obtain 10000 more logarithms, so that we have in all, besides the 30000 logarithms at first computed, those of all the numbers from 45000 down to 10000. Again, to the logarithms of all the numbers between 30000 and 27500 add log. 2, and there will thus be obtained the logarithms of all the even numbers between 60000 and 45000, also to the logarithms of every other number between 20000 and 15000 add log.

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* Or, rather, let log. 9 be determined as in (4), and then log. 90000 by adding 4 to the index.

3, and there will result the logarithm of every third odd number between 60000 and 45000, and therefore between these limits there will still remain 5000 logarithms to be determined.

3. Compute these 5000 logarithms by means of the formula (13), at art. (9), and we shall then have the logarithms of all numbers from 90000 to 10000, and, by adding log. 2 to each of those between 45000 and 50000, we shall get the logarithms of all the even numbers from 90000 to 100000, and the logarithm of every odd number will be found by taking half the sum of the logarithms of the two even numbers between which it stands, and adding 2 to the eleventh decimal in each result,* (see art. 7.)

The 100000 logarithms thus computed will be true to at least ten places of decimals, provided the 30000 first computed are true to eleven places.

(18.) When of any series of consecutive logarithms all but one are known, the unknown logarithm may be readily found, to a certain number of decimals, by means of the several differences expressed in (12), and the greater the number of terms in the proposed series, the greater will be the number of decimals that may be determined for the required logarithm. Thus, suppose the logarithms of the numbers 101, 102, 103, 104, and 105 were proposed, and that of these all were known except log. 103, then we know, from (13), that when n is about 100, the fourth difference will have no significant figure in the first seven places of decimals. Hence, referring to the expression in (13) for the fourth difference, we may obviously determine

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* For, between the limits $p = 90000$, and $p = 100000$, we shall obviously always have

$$\frac{\frac{1}{2}M}{(p+1)^2} = \cdot 00000000002.$$

the logarithm sought to seven places of decimals, by means of the equation

$$a - 4b + 6c - 4d + e = 0 \text{ or } c = \frac{4(b+d) - (a+e)}{6},$$

where

$$\begin{array}{rl}
a = \text{log. } 101 = & 2 \cdot 0043214 \\
b = \text{log. } 102 = & 2 \cdot 0086002 \\
d = \text{log. } 104 = & 2 \cdot 0170333 \\
e = \text{log. } 105 = & \underline{2 \cdot 0211893} \\
\therefore 4(b+d) = & 16 \cdot 1025340 \\
a + e = & \underline{4 \cdot 0255107} \\
& \underline{6)12 \cdot 0770233} \\
c = \text{log. } 103 = & \underline{2 \cdot 0128372}
\end{array}$$

and this will sometimes be a convenient way of determining the logarithms of the prime numbers, in certain parts of the table, or of verifying those computed by other means. If we had had a greater number of logarithms given, we might have equated the fifth or sixth difference with 0, instead of the fourth, and have then obtained a greater number of true decimals.

The expression

$$l = a + m\,\Delta^1 - \frac{m(m-1)}{2}\,\Delta^2 + \frac{m(m-1)(m-2)}{2 \cdot 3}\,\Delta^3 - \frac{m(m-1)(m-2)(m-3)}{2 \cdot 3 \cdot 4}\,\Delta^4 + \&c.$$

deduced in article (14), will enable us to calculate any term in the series from which the several differences Δ^1, Δ^2, &c. are deduced, provided we know these differences, and the first term a; but it will not enable us to insert between the terms of the proposed series other terms, so that all may follow the same law. To do this requires the aid of a more general formula, in which that above is comprehended, and which, in order to render this chapter the more complete, we here propose to investigate.

(19.) The problem may be enunciated as follows, viz.

To determine the general relation which exists between two variable quantities, x and y, from knowing the relation which exists in the particular cases, $x = \alpha$, $y = \beta$; $x = \alpha''$, $y = \beta'$; $x = \alpha''$, $y = \beta''$; &c.

Let the general relation be

$$y = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3 + \&c.$$

A, B, C, D, &c. being undeterminate coefficients; then, from the given conditions, we must have the equations

$$\left.\begin{array}{l}
\beta \ \ \ = \mathrm{A} + \mathrm{B}\alpha \ \ \ + \mathrm{C}\alpha^2 \ \ \ + \mathrm{D}\alpha^3 \ \ \ + \&c. \\
\beta' \ \ = \mathrm{A} + \mathrm{B}\alpha' \ \ + \mathrm{C}\alpha'^2 \ \ + \mathrm{D}\alpha'^3 \ \ + \&c. \\
\beta'' \ = \mathrm{A} + \mathrm{B}\alpha'' \ + \mathrm{C}\alpha''^2 \ + \mathrm{D}\alpha''^3 \ + \&c. \\
\beta''' = \mathrm{A} + \mathrm{B}\alpha''' + \mathrm{C}\alpha'''^2 + \mathrm{D}\alpha'''^3 + \&c.
\end{array}\right\} \cdots (1,)$$

$$\&c. \qquad \qquad \&c.$$

by means of which the coefficients must be determined. To do this in the simplest manner, let each equation be subtracted from the following, and we shall have these results

$$\begin{array}{l}
(\beta' - \beta) = \mathrm{B}(\alpha' - \alpha) + \mathrm{C}(\alpha'^2 - \alpha^2) + \mathrm{D}(\alpha'^3 - \alpha^3) + \&c. \\
(\beta'' - \beta') = \mathrm{B}(\alpha'' - \alpha') + \mathrm{C}(\alpha''^2 - \alpha'^2) + \mathrm{D}(\alpha''^3 - \alpha'^3) + \&c. \\
(\beta''' - \beta'') = \mathrm{B}(\alpha''' - \alpha'') + \mathrm{C}(\alpha'''^2 - \alpha''^2) + \mathrm{D}(\alpha'''^3 - \alpha''^3) + \&c.
\end{array}$$

$$\&c. \qquad \qquad \&c.$$

and by division

$$\left.\begin{array}{l}
\dfrac{\beta' - \beta}{\alpha' - \alpha} = \mathrm{B} + \mathrm{C}(\alpha' + \alpha) + \mathrm{D}(\alpha'^2 + \alpha'\alpha + \alpha^2) + \&c. = a \\
\dfrac{\beta'' - \beta'}{\alpha'' - \alpha'} = \mathrm{B} + \mathrm{C}(\alpha'' + \alpha') + \mathrm{D}(\alpha''^2 + \alpha''\alpha' + \alpha'^2) + \&c. = a' \\
\dfrac{\beta''' - \beta''}{\alpha''' - \alpha''} = \mathrm{B} + \mathrm{C}(\alpha''' + \alpha'') + \mathrm{D}(\alpha'''^2 + \alpha'''\alpha'' + \alpha''^2) + \&c. = a''
\end{array}\right\} (2,)$$

$$\&c. \qquad \qquad \&c. \qquad \qquad \&c.$$

the values of a, a', a'', &c. are known, because the first member of each equation is known.

Now these equations are of the same form as the equations (1), and A is found eliminated; hence, by performing a similar process, with the equations (2) as with the equations (1) we shall have

$$\left.\begin{array}{l} \dfrac{a' - a}{\alpha'' - \alpha'} = \mathrm{C} + \mathrm{D}\,(\alpha'' + \alpha' + \alpha) + \&\mathrm{c.} = b \\ \dfrac{a'' - a'}{\alpha''' - \alpha'} = \mathrm{C} + \mathrm{D}\,(\alpha''' + \alpha'' + \alpha') + \&\mathrm{c.} = b' \end{array}\right\} \ldots .(3,)$$

$$\&\mathrm{c.} \qquad\qquad \&\mathrm{c.} \qquad\qquad \&\mathrm{c.}$$

the values of b, b', &c. are known, because the first member of each equation is known, and B is eliminated. By continuing this process, we shall eliminate the coefficients one by one, till we come to the last, the value of which may then be determined from the final simple equation, and thence all the others, or, at least, as many as there are given conditions, so that we shall, at all events, obtain an approximation to the general relation sought, which will be so much the nearer the truth as the given relations are greater in number. Suppose, for example, only three relations are given, then the approximate general relation is

$$y = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2,$$

and equations (2) and (3) become

$$\frac{\beta' - \beta}{\alpha' - \alpha} = \mathrm{B} + \mathrm{C}\,(\alpha' + \alpha) = a$$

$$\frac{\beta'' - \beta'}{\alpha'' - \alpha'} = \mathrm{B} + \mathrm{C}\,(\alpha'' + \alpha') = a'$$

$$\frac{a' - a}{\alpha'' - \alpha} = \mathrm{C} = b.$$

substituting this value of C, in equation (2,) we have

$$\mathrm{B} = a - b\,(\alpha' + \alpha.)$$

Also, since from the proposed equation we have

$$\beta = \mathrm{A} + \mathrm{B}\alpha + \mathrm{C}\alpha^2,$$

we obtain for A, after having replaced B and C by the values just determined, the expression

$$\mathrm{A} = \beta - a\alpha + b\alpha\alpha'.$$

Having thus determined the values of the three coefficients, we have, for the general relation sought, this approximation:

$$y = \beta - a\alpha + b\alpha\alpha' + (a - b\alpha' - b\alpha)\,x + bx^2,$$

or

$$y = \beta + a\,(x - \alpha) + b\,(x - \alpha)\,(x - \alpha').$$

If four conditions had been given, we should have obtained the expression

$$y = \beta + a\,(x - \alpha) + b\,(x - \alpha)\,(x - \alpha') + c\,(x - \alpha)\,(x - \alpha')\,(x - \alpha'') \dots (4,)$$

and so on. This is Newton's method of solution; but, observes Lagrange, it may be much more simply obtained from the following considerations:

Since y ought to become β, β', β'', &c. when x becomes $\alpha, \alpha', \alpha''$ &c. it is obvious that the expression for y must be of the form

$$y = \mathrm{A}'\beta + \mathrm{B}'\beta' + \mathrm{C}'\beta'' + \&\text{c.}$$

where the quantities A′, B′, C′, &c. must be functions of x, such that, when we put

$$\begin{array}{ll} x = \alpha, \text{ we must have} & \mathrm{A}' = 1, \mathrm{B}' = 0, \mathrm{C}' = 0, \&\text{c.} \\ x = \alpha', & \mathrm{A}' = 0, \mathrm{B}' = 1, \mathrm{C}' = 0, \&\text{c.} \\ x = \alpha'', & \mathrm{A}' = 0, \mathrm{B}' = 0, \mathrm{C}' = 1, \&\text{c.} \\ \&\text{c.} & \&\text{c.} \end{array}$$

consequently the values of A′, B′, C′, &c. must necessarily take the form

$$\mathrm{A}' = \frac{(x - \alpha')\,(x - \alpha'')\,(x - \alpha''')\ \&\text{c.}}{(\alpha - \alpha')\,(\alpha - \alpha'')\,(\alpha - \alpha''')\ \&\text{c.}}$$

$$\mathrm{B}' = \frac{(x - \alpha)\,(x - \alpha'')\,(x - \alpha''')\ \&\text{c.}}{(\alpha' - \alpha)\,(\alpha' - \alpha'')\,(\alpha' - \alpha''')\ \&\text{c.}}$$

$$\mathrm{C}' = \frac{(x - \alpha)\,(x - \alpha')\,(x - \alpha''')\ \&\text{c.}}{(\alpha'' - \alpha)\,(\alpha'' - \alpha')\,(\alpha'' - \alpha''')\ \&\text{c.}}$$

where the number of factors in each numerator and denominator is one less than the number of given conditions. Hence the general expression for y is

$$\begin{aligned} y = {} & \frac{(x - \alpha')\,(x - \alpha'')\,(x - \alpha''')\ \&\text{c.}}{(\alpha - \alpha')\,(\alpha - \alpha'')\,(\alpha - \alpha''')\ \&\text{c.}}\,\beta \\ & + \frac{(x - \alpha)\,(x - \alpha'')\,(x - \alpha''')\ \&\text{c.}}{(\alpha' - \alpha)\,(\alpha' - \alpha'')\,(\alpha' - \alpha''')\ \&\text{c.}}\,\beta' \\ & + \frac{(x - \alpha)\,(x - \alpha')\,(x - \alpha''')\ \&\text{c.}}{(\alpha'' - \alpha)\,(\alpha'' - \alpha')\,(\alpha'' - \alpha''')\ \&\text{c.}}\,\beta'' \end{aligned}$$

This expression differs from that of Newton only in form; for if, in Newton's formula, the values of a, b, c, &c. be developed, and the results arranged according to the quantities β, β', β'', &c. we shall obtain the same expression as that deduced by Lagrange.

(20.) When the given quantities $\alpha, \alpha', \alpha''$, &c. are in arithmetical progression, the expression (4) leads to a formula of extensive use in interpolations.

Let h be the common difference in the progression, then

$$\alpha' = \alpha + h,\ \alpha'' = \alpha + 2h,\ \alpha''' = \alpha + 3h, \text{ \&c.}$$

let, also, $x = \alpha + h'$, then

$$x - \alpha = h',\ x - \alpha' = h' - h,\ x - \alpha'' = h' - 2h, \text{ \&c.}$$

Now, putting $\Delta'\beta, \Delta'\beta', \Delta'\beta''$, &c. for the several first differences, $\beta' - \beta, \beta'' - \beta', \beta''' - \beta''$, &c. we have (2)

$$a = \frac{\Delta'\beta}{h},\ a' = \frac{\Delta'\beta'}{h},\ a'' = \frac{\Delta'\beta''}{h}, \text{ \&c.}$$

putting, in like manner, $\Delta^2\beta, \Delta^2\beta'$, &c. for the second differences, $\Delta'\beta' - \Delta'\beta, \Delta'\beta'' - \Delta'\beta'$, &c, we have (3)

$$b = \frac{\Delta^2\beta}{1 \cdot 2h^2},\ b = \frac{\Delta^2\beta'}{1 \cdot 2h^2}, \text{ \&c.}$$

substituting, also, $\Delta^3\beta$, &c. for the third differences, $\Delta^2\beta' - \Delta^2\beta$, &c. there results

$$c = \frac{\Delta^3\beta}{1 \cdot 2 \cdot 3h^3}, \text{ \&c.}$$

therefore the formula (4) becomes

$$y = \beta + \frac{h'}{h}\Delta'\beta + \frac{h'(h'-h)}{h \cdot 2h}\Delta^2\beta + \frac{h'(h'-h)(h'-2h)}{h \cdot 2h \cdot 3h}\Delta^3\beta + \text{ \&c.} \ldots . (5.)$$

and this, when $h = 1$, becomes identical to the formula already given.

As an example of the application of this formula, let there be given log. 3100, log. 3110, log. 3120, to find the eighteen intermediate logarithms to seven or eight decimal places.

log. 3117 = 3 · 4937368025
1393075
log. 3118 = 3 · 4938761100
1392625
log. 3119 = 3 · 4940153725
1392175
log. 3120 = 3 · 4941545900

This last result proves the accuracy of all the preceding operations. Indeed the logarithms here computed are all true to the nearest unit, as far as eight decimals, thus possessing the same accuracy as the logarithms with which we set out. Hence, when the first 3000 logarithms are previously obtained, to about eight decimals each, this easy process may be advantageously employed.

We shall give one more example of the application of the general formula (5).

To compute the logarithm of the number 3 · 1415926536, by means of a table of logarithms of numbers from 1 to 1000, calculated to ten places of decimals.

Taking from the given table the logarithms of the numbers 3 · 14, 3 · 15, 3 · 16, 3 · 17, and 3 · 18, we have the following table of differences, those after the fourth being neglected, because they are too small to affect the accuracy of the tenth decimal in the logarithm to be computed :

Numbers.	Log.	Δ	Δ^2	Δ^3	Δ^4
3 · 14	·4969296481	·0013809057	.0000043769	·0000000277	·0000000003
3 · 15	·4983105538	·0013765288	.0000043492	·0000000274	
3 · 16	·4996870826	·0013721796	.0000043218		
3 · 17	·5010592622	·0013678578			
3 · 18	·5024271200				

Now $3 \cdot 14$ being taken for a, and $3 \cdot 1415926536$ for x, we have $h' = x - a = \cdot 0015926536$; hence

$$\frac{h'}{h} = \cdot 15926536$$

$$\frac{h'-h}{2h} = \frac{h'}{2h} - \tfrac{1}{2} = - \cdot 42036732$$

$$\frac{h'-2h}{3h} = \frac{h'}{3h} - \tfrac{2}{3} = - \cdot 61357821$$

$$\frac{h'-3h}{4h} = \frac{h'}{4h} - \tfrac{3}{4} = - \cdot 71018366$$

These values substituted in the general formula (5) give

$$y = \log. 3 \cdot 1415926536 = \cdot 4971498726.$$

CHAPTER III.

ON THE DIRECT COMPUTATION OF LOGARITHMS INDEPENDENTLY OF EACH OTHER.

(21.) To compute the logarithm of any proposed number without the aid of other logarithms, was considered, by the earlier computers, to be a problem of the first importance in the formation of a table. Methods for performing this computation were given by Napier; of these, that which Briggs prefers, and which appears the easiest, consists in taking a great number of continued geometrical means between 1 and the given number whose logarithm is required; that is, first extracting the square root of the given number, then the root of the first root, the root of the second root, the root of the third root, and so on, till the last root shall exceed 1 by a very small decimal, so small, indeed, that the 1 may be followed by half as many ciphers as there are to be true decimal places in the logarithm sought. The decimal part of this last root will be the logarithm of the root itself, to the proposed number of places, and therefore, by multiplying it by such a power of 2 as is denoted by the number of extractions, the sought logarithm will be obtained. But although Mr. Briggs devises some ingenious contrivances to abridge the above labour of extractions, the process is at best very tedious. We shall, therefore, not dwell upon these or other contrivances adopted in the infancy of this science to simplify the problem in question, but shall refer the inquiring reader for the most copious information of this kind to Dr. Hutton's learned History of Logarithms. We shall here content ourselves with presenting to the student what seems to us to be the easiest method of computing logarithms, by a direct and independent process. It is due to Mr. Manning, and is published in the Philosophical Transactions for 1806, and as the paper is short, and popularly written, we shall extract it nearly as it stands.

(22.) This method is, as Mr. Manning observes, peculiarly easy of application, requiring no division, multiplication, or extraction of roots, and has its relative advantages highly increased by increasing the number of decimal places to which the computation is carried.

The chief part of the working consists in merely setting down a number under itself removed one or more places to the right, and subtracting and repeating this operation; and consequently is very little liable to mistake. Moreover, from the commodious manner in which the work stands, it may be revised with extreme rapidity. It may be performed after a few minutes' instruction by any one who is competent to subtract. It is as easy for large numbers as for small; and, on an average, about 27 subtractions will furnish a logarithm accurately to about 10 places of decimals. In general, $9 \times \frac{n+1}{2}$ subtractions will be accurate to $2n$ places of decimals. In computing hyperbolic logarithms by this method, it is necessary to have previously established the h logs. of $\frac{10}{9}$, $\frac{100}{99}$, $\frac{1000}{999}$, &c. of 2 and of 10.

With respect to the logs.′ of $\frac{10}{9}$, $\frac{100}{99}$, $\frac{1000}{999}$, &c. their computation is very easy, they being the respective sums of the following series, (see art. 4,)

$$\frac{1}{10} + \frac{1}{2 \cdot 10^2} + \frac{1}{3 \cdot 10^3} + \frac{1}{4 \cdot 10^4} + \&c.$$
$$\frac{1}{100} + \frac{1}{2 \cdot 100^2} + \frac{1}{3 \cdot 100^3} + \frac{1}{4 \cdot 100^4} + \&c.$$
$$\frac{1}{1000} + \frac{1}{2 \cdot 1000^2} + \frac{1}{3 \cdot 1000^3} + \frac{1}{4 \cdot 1000^4} + \&c.$$

which series are very expeditiously summed by the directions given in the article just referred to. As to the log.′ of 2 and 10, we have already explained (2, 4,) easy methods of determining them; we shall, however, presently exhibit the process of finding them from the direct method which we are about to explain. As, in this method, there will be frequent occasion for the logs. of $\frac{10}{9}$, $\frac{100}{99}$, &c. of 10 and of 2, as well as for the modulus M. Mr. Manning gives the following tables of these numbers, and of their multiples, as far as 9, in order to save the labour of multiplications:

log.′ $\frac{10}{9}$ $= \cdot 105360515655$
log.′ $\frac{100}{99}$ $= \cdot 010050335853$
log.′ $\frac{1000}{999}$ $= \cdot 001000500333$
log.′ $\frac{10000}{9999}$ $= \cdot 000100005$
log.′ $\frac{100000}{99999}$ $= \cdot 00001000005$
log.′ $\frac{1000000}{999999}$ $= \cdot 000001$
log.′ $\frac{10000000}{9999999}$ $= \cdot 0000001$
log.′ $\frac{100000000}{99999999}$ $= \cdot 00000001$
&c. &c.
log.′ 2 $= \cdot 693147180637$
log.′ 10 $= 2 \cdot 302585093217$
M $= \cdot 434294481861$

Multiples of the above numbers.

Double log.′ $\frac{10}{9}$ = ·210721031310
triple = ·316081546965
quadruple = ·421442062620
5 ple = ·526802578275
6 ple = ·632163093930
7 ple = ·737523609585
8 ple = ·842884125240
9 ple = ·948244640895

Double log.′ $\frac{100}{99}$ = ·020100671706
triple = ·030151007559
quadruple = ·040201343412
5 ple = ·050251679265
6 ple = ·060302015118
7 ple = ·070352350971
8 ple = ·080402686824
9 ple = ·090453022677

Double log.′ $\frac{1000}{999}$ = ·002001000666
triple = ·003001500999
quadruple = ·004002001332
5 ple = ·005002501665
6 ple = ·006003001998
7 ple = ·007003502331
8 ple = ·008004002664
9 ple = ·009004502997

Double log.′ 2 = 1·386294361274
triple = 2·079441541911
quadruple = 2·772588722548
5 ple = 3·465735903185
6 ple = 4·158883083822
7 ple = 4·852030264459
8 ple = 5·545177445096
9 ple = 6·238324625733

Double log.′ 10 = 4 · 605170186434
triple = 6 · 907755279651
quadruple = 9 · 210340372868
5 ple = 11 · 512925466085
6 ple = 13 · 815510559302
7 ple = 16 · 118095652519
8 ple = 18 · 420680745736
9 ple = 20 · 723265838953

Double M = 868588963722
triple = 1 · 302883445583
quadruple = 1 · 737177927444
5 ple = 2 · 171472409305
6 ple = 2 · 605766891166
7 ple = 3 · 040061373027
8 ple = 3 · 474355854888
9 ple = 3 · 908650336749

We now proceed to the method itself.

(23.) To find the log.′ of any number not exceeding 2.

Rule. Set the number under itself to be subtracted from itself, but removed so many places to the right as shall be necessary to make the remainder greater than 1; subtract. Proceed in the same manner with the remainder, and so on, till the remainder becomes 1, followed by half as many ciphers as the number of decimals you work to; suppose at the end of the operation you find that you have removed *one* place to the right and subtracted b times *two* places, c times *three* places, d times, &c.; then b log.′ $\frac{10}{9}$ + c log.′ $\frac{100}{99}$ + d log.′ $\frac{1000}{999}$ + &c. + decimal part of the last remainder is the log.′ sought. And these numbers are collected out of the preceding table, for b, c, d, &c. can none of them ever exceed 9.

EXAMPLE I.

To find log.′ 2 to 10 places of decimals.

```
2
 ·2
————
1 · 8
    18
————
1 · 62
    162
————
1 · 458
    1458
————
1 · 3122
    13122
————
1 · 18098
    118098
————
1 · 062882 . . . . (6)
     1062882
————
1 · 05225318
     105225318
————
1 · 0417306482
     10417306482
————
1 · 031313341718
     103131334171
————
1 · 0210002083009
     102100020830
————
1 · 0107902062179
     101079020621
————
1 · 0006823042558 . . . . (6)
         1000682304
————
1 · 0005822360254
         1000582236
————
```

1 · 0004821778018
1000482177

1 · 0003821295841
1000382129

1 · 0002820913712
1000282091

1 · 0001820631621
1000182063

1 · 0000820449558 . . . . (6)
100008204

1 · 0000720441354
100007204

1 · 0000620434150
100006204

1 · 0000520427946
100005204

1 · 0000420422742
100004204

1 · 0000320418538
100003204

1 · 0000220415334
100002204

1 · 0000120413130
100001204

1 · 0000020411926 . . . . (8)
10000020

1 · 0000010411906
10000010

1 · 0000000411896 remainder.

6 log.′ $\frac{10}{9}$ = ·632163093930
6 log.′ $\frac{100}{99}$ = ·060302015118
6 log.′ $\frac{10000}{9999}$ = ·00060003
8 log.′ $\frac{100000}{99999}$ = ·0000800004
2 log.′ $\frac{1000000}{999999}$ = ·000002
dec. of last rem. = ·000000041189

log.′ 2[*] = ·693147180637

EXAMPLE II.

To find log.′ 1 · 25

1 · 25
125

1 · 125
1125

1 · 0125 . . . (2)
10125

1 · 002375 . . . . (1)
1002375

1 · 001372625
1001372625

1 · 000371252375 . . . . (2)
1000371252

1 · 0002712152498
1000271215

1 · 0001711881283
1000171188

1 · 0000711710095 . . . . (3)
100007117

1 · 0000611702978
100006117

---

[*] If the common logarithm is sought, we must multiply the modulus by this result, or add together the proper multiples of the modulus taken out of the foregoing table.

$$\begin{array}{r}
1 \cdot 0000511696861 \\
100005116 \\
\hline
1 \cdot 0000411691745 \\
100004116 \\
\hline
1 \cdot 0000311687629 \\
100003116 \\
\hline
1 \cdot 0000211684513 \\
100002116 \\
\hline
1 \cdot 0000111682397 \\
100001116 \\
\hline
1 \cdot 0000011681281 \ldots\ldots (7) \\
10000011 \\
\hline
\underline{1 \cdot 0000001681270} \text{ rem.}
\end{array}$$

$$\begin{array}{rl}
2 \text{ log.}' \tfrac{10}{9} & = \cdot 210721031310 \\
1 \text{ log.}' \tfrac{100}{99} & = \cdot 010050335853 \\
2 \text{ log.}' \tfrac{1000}{999} & = \cdot 002001000666 \\
3 \text{ log.}' \tfrac{10000}{9999} & = \cdot 000300015000 \\
7 \text{ log.}' \tfrac{100000}{99999} & = \cdot 000070000350 \\
1 \text{ log.}' \tfrac{1000000}{999999} & = \cdot 000001 \\
\text{decimal of last rem.} & = \cdot 000000168127 \\
\hline
\text{log.}' 1 \cdot 25 & = \cdot \underline{223143551306}
\end{array}$$

(24.) To find the log.′ of any number whole or mixed.

Rule. Reduce the given number (if necessary) to a whole or mixed number less than 2, by setting the decimal point after the first significant figure, or if the given number be 10, or a power of 10 after the first 0, and then dividing by 2 (if necessary) till the integral part is 1.*

Find the log.′ of this reduced number by the last rule, and add to it or subtract from it as many times log.′ 10 as the decimal point was removed places to the left or right; also add to it as many times log.′ 2 as there were divisions by 2. The sum is the log.′ required.

---

* Three divisions by 2 will always suffice.

EXAMPLE III.

To find log.' 10.

By last example,

$$\log.' 1\cdot25 = \log.\frac{10}{2^3} = \quad 223142551306$$

$$\log. 2^3 = 2\cdot079441541911$$

$$\underline{\overline{2\cdot302585093217}}$$

EXAMPLE IV.

To find log. 7 to 5 places of decimals

4)7
1·75
175
1·575
1575
1·4175
14175
1·27575
127575
1·148175
1148175
1·0333575 . . . . (5)
1033357
1·02302303
1023023
1·01279370
1012793
1·00266577 . . . . (3)
100266
1·00166311
100166
1·00066145 = last rem.

$$\begin{array}{rr}
5\ \log.'\ \frac{10}{9} = & 5268025 \\
3\ \log.'\ \frac{100}{99} = & 0301510 \\
2\ \log.'\ \frac{1000}{999} = & 0020010 \\
\text{Dec. of rem.} = & 0006614 \\
2\ \log.'\ 2 = & 1 \cdot 3862943 \\
\hline
\log.'\ 7 = & 1 \cdot 9459102 \\
\hline
\end{array}$$

From the small number of subtractions log.′ 7 must be true to six places of decimals.

EXAMPLE V.

To find log.′ 5548748 to 6 places of decimals

$$\begin{array}{r}
4)5 \cdot 548748 \\
\hline
1 \cdot 387187 \\
\cdot 1387187 \\
\hline
1 \cdot 2484683 \\
12484683 \\
\hline
1 \cdot 12362147 \\
112362147 \\
\hline
1 \cdot 011259323 \\
10112593 \\
\hline
1 \cdot 001146730 \\
1001146 \\
\hline
1 \cdot 000145584 \\
\hline
\end{array}$$

$$\begin{array}{rr}
3\ \log.'\ \frac{10}{9} = & 316081546 \\
1\ \log.'\ \frac{100}{99} = & 010050335 \\
1\ \log.'\ \frac{1000}{999} = & 001000500 \\
\text{Dec. of rem.} = & 000145584 \\
\hline
\log.'\ 1 \cdot 387187 = & 327277965 \\
6\ \log.'\ 10 = & 13 \cdot 815510559 \\
2\ \log.'\ 2 = & 1 \cdot 386294361 \\
\hline
\log.'\ 5548748 = & 15 \cdot 529082885 \\
\hline
\end{array}$$

(25.) The demonstration of the above rules is obvious. Setting the figures of a number one place to the right is dividing that number by 10, two places by 100, three places by 1000, &c. And subtracting a number so placed from the number itself is subtracting a 10th, a 100th, &c. of the number itself, and, consequently, the results are respectively $\frac{9}{10}$ths, $\frac{99}{100}$ths, &c. of the numbers proposed. Let b, c, d, &c. denote as in the rule, then the original number $= (\frac{10}{9})^b + (\frac{100}{99})^c + (\frac{1000}{999})^d \times$ &c. $\times$ last remainder. Therefore the log.′ of the original number $= b$ log.′ $\frac{10}{9} + c$ log.′ $\frac{100}{99} + d$ log.′ $\frac{1000}{999}$ + &c. + log.′ last rem. Now the last remainder being unity followed by a certain number of decimal ciphers, its correct log.′, as far as twice that number of places, is (as is well known)* the decimal part itself of that remainder. Hence the value is manifest.

(26.) The preceding method of finding the logarithm of any number may be often conveniently applied by way of verification, when we have any reason to doubt the accuracy of any particular logarithm, computed by a different process; and it is for this purpose principally that the method is valuable.

CHAPTER IV.

MISCELLANEOUS PARTICULARS RESPECTING LOGARITHMS.

(27.) Having, in the preceding chapters, given tolerably copious directions for computing a table of logarithms of all numbers from 1 to 100000, we shall now throw together a few miscellaneous particulars that ought to have a place in an essay on this subject.

---

* This will at once appear from referring to the series (1) in representing the decimal part above mentioned, for n^2 will be a decimal, having at least twice as many ciphers prefixed as n has.

1. To extend a table of logarithms to numbers beyond 100000 by using only the logarithms of numbers below 1000.

This may be easily done by means of the formula

$$\log.(m+n) = \log. m + 2\mathrm{M}\left\{\frac{n}{2m+n} + \tfrac{1}{3}\left(\frac{n}{2m+n}\right)^3 + \tfrac{1}{5}\left(\frac{n}{2m+n}\right)^5 + \&c.\right\}$$

by which the table may be extended as required. For example, suppose log. 543839 is required; then, since $543839 = 543834 + 5 = 729 + 746 + 5$, we have, by this formula,

$$\log. 543839 = \log. 729 + \log. 746 + 2\mathrm{M}\left\{\frac{5}{1087673} + \tfrac{1}{3}\left(\frac{5}{1087673}\right)^3 + \&c.\right\}$$

The second term of this series cannot have a significant figure before the fifteenth decimal, and may, therefore, be omitted. If the logarithms of numbers below 1000 be calculated to 11 decimals, all those between 100000 and 1000000 may in this way be computed to at least ten decimals.

2. The logarithms of negative numbers are excluded from all systems having positive bases.

At the commencement of this essay, we assumed a to be the base of any system of logarithms, and $1+n$ to be any proposed number, and we proceeded to determine what value x ought to have to satisfy the equation $a^x = 1 + n$; such a value we called the logarithm of the number $1+n$, and we found for it this expression, viz.

$$x = \log.(1+n) = \mathrm{M}\left(n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \&c.\right)$$

But this fails to represent the true value of x, whenever $1+n$ is a negative number, provided a be positive, and this failure we ought to expect; for a being of arbitrary value, we might assume it such that for any given negative number $1 + n$ the equation $a^x = 1 + n$ should be absolutely impossible. Thus, suppose $n + 1$ were -2, we have then only to make a equal to 2, or 8, or 32, or, indeed, to any odd power of 2, in order to render the above equation impossible, and generally for any negative number $-p$ the value of x will be impossible, provided a be any odd power of p. On this account all negative values of $n + 1$ are necessarily excluded from the foregoing expression, which we see is always possible whatever be the value of $n + 1$. It has been said that because $a^{\frac{1}{2}}$ may be either a positive or a negative number, therefore $\frac{1}{2}$ must be equally the logarithm, both of the positive and of the negative number; but this is not true, inasmuch as $a^{\frac{1}{2}}$ is not *both* a positive and a negative number, although it may be *either* the one or the other. If $\frac{1}{2}$ be admitted to represent the logarithm of the positive value of $a^{\frac{1}{2}}$, the same value taken negatively can have no logarithm in that system, for if it have any logarithm, that logarithm must be $\frac{1}{2}$, yet, as observed above, both values cannot have this logarithm.

3. A system of logarithms cannot be formed from a negative base. For all odd powers and roots of such a base would be negative; hence those numbers taken equal to any of these and positive, would have no logarithms in the system, and since all the even powers of the base are positive, the same numbers taken negatively would have no logarithms, and moreover all those numbers which are even roots of the base taken positively, would be without logarithms in such a system, whether these roots be considered as positive or negative numbers.

4. In any system whose base is greater than 1, the logarithm of an infinite number is infinitely positive, and the logarithm of 0 is infinitely negatively; but, if the base

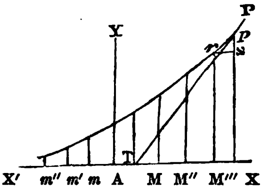
be less than 1, then, on the contrary, the logarithm of an infinite number is infinitely negative, and the logarithm of 0 infinitely positive. This is at once obvious; for, in the equation $a^x = \mathrm{N}$, if a exceed 1, x must increase as long as N does, and become infinite with it. If N be less than 1, x must be negative, that is, $\frac{1}{a^x} = \mathrm{N}$, and the first side of this equation can become 0 only when x is infinite. But when a is less than 1, a^x becomes 0 when x is infinite, and $\frac{1}{a^x}$ becomes infinite in like circumstances.

If, in the series (4), p. 3, we suppose $p = 1$, it becomes

$$\log. \tfrac{1}{0} = 1\mathrm{M}\,(1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \&\text{c.})$$

but $\frac{1}{0}$ is infinite; therefore log. $\frac{1}{0}$ is infinite, consequently the sum of the reciprocals of the arithmetical series of odd numbers is infinite. It follows, therefore, that the series $\frac{1}{1} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ &c. which, besides the above, contains also the terms $\frac{1}{2}$, $\frac{1}{4}$, &c. must be infinite.

5. We may give a geometrical representation to numbers and their logarithms. Thus, take any point, A, upon the indefinite straight line X′, X, and draw the perpendicular, AY; then if from A there be set off on X′X, both ways an indefinite number of lengths or abscissas, AM, AM′, AM″, &c. and Am, Am', Am'', &c. and from their extremities parallels to AY be drawn; and if, moreover, we limit these parallels, or ordinates, so that, calling any abscissa x, its ordinate, y, may always be equal to a^x, a being an assumed number or length, the ordinates will obviously represent those numbers for which the abscissas represent the logarithms.



The curve, passing through the extremities of the ordinates, is called the *lagarithmic curve*, and its equation is $y = a^x$. At the point A, $x = 0$, therefore $y = a^0 = 1$, that is, the logarithm of 1 is 0, and if $AM = 1$, then $y = M\,P = a$, showing that the logarithm of the base is 1. If this assumed base exceed unity, the ordinates of the curve increase as the abscissas to the right of A, that is, as the positive abscissas, increase, but they diminish as the negative abscissas, or those to the left, increase, for the general value of the ordinates on this side is $y = \frac{1}{a^x}$; hence, the ordinates increase till they become infinitely long to the right of A, and diminish till they become infinitely short to the left of A. The contrary will obviously have place if a be less than unity.

If any abscissa be taken, so as to represent any even fractional number, as ½, ¼, &c. the position of the corresponding ordinate would at first appear ambiguous, or we might suppose that the logarithmic curve admits of both positive and negative ordinates to the same abscissas; but, by increasing or diminishing the value of x, in this case, by a quantity infinitely small, we find that the ordinates adjacent to this cannot be negative, so that the negative ordinate that seemed to claim our notice belongs not to the curve, but to a point out of it, and entirely unconnected with it, and this is a geometrical confirmation of what we advanced above about the logarithms of negative numbers.

But the most remarkable property of this curve is the following, viz. That at any point, P, the ordinate and tangent always intercept a portion, $M'''\,T$, of the axis, AX, of the same constant length, and this property may be proved as follows:

Infinitely near to PM''' conceive the ordinate pX to be drawn, then the infinitely small portion, Pp, of the curve may be considered as coincident with the tangent PT, so that the little triangle, Psp, will be similar to the triangle TM'''P. Now, putting $AM''' = x$, $M'''P = y$, $M'''X = Ps = i$, and $ps = k$, we have, in virtue of the general relation,

$$y = a^x \text{ or } x = \log. y$$
$$y + k = a^{x+i} \text{ or } x + i = \log. (y + k),$$

the base of the system being a. Since $\log. (y + k) = \log. y + \log. (1 + \frac{k}{y})$, we have, by the formula (1),

$$\log. (y + k) = \log. y + M \{\frac{k}{y} - \tfrac{1}{2} (\frac{k}{y})^2 + \tfrac{1}{3} (\frac{k}{y})^3 - \&c.\}$$

that is,

$$x + i = x + M \{\frac{k}{y} - \tfrac{1}{2} (\frac{k}{y})^2 + \tfrac{1}{3} (\frac{k}{y})^3 - \&c.\}$$

therefore

$$\frac{i}{k} = \frac{M}{y} - M \{\tfrac{1}{2} \frac{k}{y^2} - \tfrac{1}{3} \frac{k^2}{y^3} + \&c.\}$$

The second member of this equation expresses the value of the ratio $\frac{i}{k}$, when i and k are infinitely small, or, more strictly speaking, it expresses the accurate value of this ratio, when i and k vanish, or become each $= 0$, but then the expression within the brackets must at the same time vanish; hence

$$\frac{i}{k} = \frac{M}{y} = \frac{M}{M'''P};$$

but, on account of the similar triangles, $\frac{i}{k} = \frac{M'''T}{M'''P}$, therefore

$$\frac{M'''T}{M'''P} = \frac{M}{M'''P} \therefore M'''T = M,$$

so that the *subtangent*, as the line M'''T is called, is constantly equal to that quantity which Cotes called the ***modulus*** of the system.

Numbers and their logarithms may also be represented by means of other curves, particularly by the hyperbolic arc, and its asymptotes for the areas included between the curve and one asymptote, when they are bounded by ordinates parallel to the other asymptote, are analogous to the logarithms of their abscissas, or parts of the first asymptote, and the modulus of the system represented will be expressed by the sine of the angle of inclination of the asymptotes,* so that, when the hyperbola is equilateral, that is, when the asymptotes are perpendicular to each other, the modulus must be equal to radius, and if this be represented by 1, the system will be the Napierian system, more frequently called the *hyperbolic system,* although this is a less definite designation, since, as just observed, any system may be represented by means of the hyperbolic arc and asymptotes. The above property of the hyperbola is demonstrated in most books which treat of the quadrature of curves.

(28.) What we have hitherto said about logarithms, both in this essay, and in our treatise on Algebra, has reference entirely to their importance in numerical operations. But logarithmic formulas may be advantageously employed to abridge many analytical inquiries in different parts of the mathematics. We shall give one instance of this in the solution of the following algebraical problem.

To find the sum of the mth powers of the roots of an algebraical equation from having the coefficients given.

---

* The *trigonometrical* sine, which is that to a radius whose numerical value is 1, is not meant here; the radius in the present case will be equal to the side of the rhombus inscribed between the curve and asymptotes, and having a vertex coinciding with that of the hyperbola.

Let the equation be

$$x^n + Ax^{n-1} + Bx^{n-2} \;.\;.\;.\;.\;.\;.\;.\;.\; + V = 0,$$

and let us represent its n roots by a, b, c, d, &c. so that we shall have

$$x^n + Ax^{n-1} + Bx^{n-2} + \&c. = (x-a)\,(x-b)\,(x-c)\,(x-d)\;\&c.$$

or, putting $\frac{1}{z}$ for x, and then multiplying both members by z^n, we have

$$1 + Az + Bz^2 \ldots + Vz^n = (1-az)\,(1-bz)\,(1-cz)\,(1-dz)\;\&c.$$

or, taking the Napierian logarithms of both sides,

$$\log.'(1 + Az + Bz^2 \;.\;.\;.\;.\; + Vz^n)$$
$$= \log.'(1-az) + \log.'(1-bz) + \log.'(1-cz) + \log.'(1-dz) + \&c.$$

Now, since

$$\log.'(1-n) = -(n + \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 + \tfrac{1}{4}n^4 + \&c.)$$

we have, by putting instead of n, az, bz, cz, dz, &c. successively, the equation

$$\begin{aligned}\log.'(1 + Az + Bx^2 \ldots + Vz^n) = &-(a + b + c + d + \&c.)\,z \\ &-(a^2 + b^2 + c^2 + d^2 + \&c.)\,\frac{z^2}{2} \\ &-(a^3 + b^3 + c^3 + d^3 + \&c.)\,\frac{z^3}{3} \\ &-(a^4 + b^4 + c^4 + d^4 + \&c.)\,\frac{z^4}{4} \\ &\quad\&c.\end{aligned}$$

or putting, for brevity,

$$\begin{aligned} S_1 &= a + b + c + d + \&c. \\ S_2 &= a^2 + b^2 + c^2 + d^2 + \&c. \\ S_3 &= a^3 + b^3 + c^3 + d^3 + \&c. \\ S_4 &= a^4 + b^4 + c^4 + d^4 + \&c. \\ \&c. & \qquad\quad \&c. \end{aligned}$$

we have

$$\log.'(1 + Az + Bz^2 + \ldots\ldots Vz^n) = -zS_1 - \tfrac{1}{2}z^2S_2 - \tfrac{1}{3}z^3S_3 - \tfrac{1}{4}z^4S_4 - \&c. \ldots.(1)$$

Now, to develope the first member of this equation, we must substitute $Az + Bz^2 + \ldots. Vz^n$ for n in the formula

$$\log.'(1+n)=n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\&c.$$

and arranging the result according to the powers of z, we get

$$-zS_1-\tfrac{1}{2}z^2S_2-\tfrac{1}{3}z^3S_3-\tfrac{1}{4}S_4-\&c.$$

$$\begin{array}{r|l|l|l}
=Az+B & z^2+C & z^3+D & z^4+\&c. \\
-\frac{A^2}{2} & -AB & -AC & \\
 & +\frac{1}{3}A^3 & -\frac{1}{2}B^2 & \\
 & & +A^2B & \\
 & & -\frac{1}{4}A^4 &
\end{array}$$

hence, equating the coefficients of the like powers of z, we get these results:

$S_1=-A$

$S_2=-2B+A^2$

$S_3=-3C+3AB-A^3$

$S_4=-4D+2(2AC+B^2)-4A^2B+A^4$

$S_5=-5E+5(AD+BC)-5(A^2C+AB^2)+5A^3B-\frac{5}{5}A^5$

&c. &c.

From these results we may easily pass to the converse problem, viz. to find the coefficients from having the sums of the powers given. But the solution of this may be very elegantly obtained, without employing these results, by the aid of the exponential theorem.* Thus the equation (1) above, viz.

$$\log.'(1+Az+Bz^2+\ldots+Vz_n)=-zS_1-\tfrac{1}{2}z^2S_2-\tfrac{1}{3}z^3S_3-\tfrac{1}{4}z^4S_4-\&c.$$

is the same as

$$\log.'(1+Az+Bz^2+\ldots+Vz^n)=\log.'e^{-zS_1-\frac{1}{2}z^2S_2-\frac{1}{3}z^3S_3-\frac{1}{4}z^4S_4-\&c.}$$

e being the base of the Napierian system; hence, passing from the logarithms to the numbers, we have

$$1+Az+Bz^2+\ldots+Vz^n=e^{-zS_1-\frac{1}{2}z^2S_2-\frac{1}{3}z^3S_3-\frac{1}{4}z^4S_4-\&c.}$$

therefore, substituting $-zS_1-\frac{1}{2}z^2S_2-\frac{1}{3}z^3S_3-\frac{1}{4}z^4S_4-\&c.$ for x, in the development

$$e^x=1+\frac{x}{1}+\frac{x^2}{1\cdot 2}+\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}+\&c.$$

---

* See the note at the end.

and arranging the result according to the powers of z, we have

$$1+Az+Bz^2+\ldots+Vz^n=1-S_,z\begin{array}{r|r|l} -\frac{S^2}{2} & z^2 - \quad \frac{S_3}{3} & z^3 - \&c. \\ +\frac{S_,^2}{2} & +\quad \frac{S_,S^2}{1\cdot 2} & \\ & -\frac{S_,^3}{1\cdot 2\cdot 3} & \end{array}$$

from which, by comparing the coefficients of the like powers of z, we get, for the sought coefficients, the expressions

$$\begin{aligned} A &= -S_, \\ B &= -\frac{S_2}{2}+\frac{S'^2}{1\cdot 2} \\ C &= -\frac{S_3}{3}+\frac{S_,S^2}{1\cdot 2}-\frac{S_,^3}{1\cdot 2\cdot 3} \\ \&c. & \qquad\qquad \&c. \end{aligned}$$

(29.) The problem which determines the sums of the powers of the roots of an equation in terms of the coefficients has been employed both by Newton and Lagrange for the purpose of approximating to the greatest root of the equation. The method proposed by Lagrange is generally preferred, although it is, in certain cases, very defective. As the defect here adverted to has, we believe, never been noticed, we shall perhaps be excused for briefly explaining Lagrange's method here, for the purpose of pointing out, and of removing, its inaccuracy.

Suppose a, b, c, &c. to be the roots of an equation and that a is the greatest, abstracting from the sign.

Then, putting

$$\begin{aligned} S_u &= a^u + b^u + c^u + \&c. \\ S_{u-1} &= a^{u-1} + b^{u-1} + c^{u-1} + \&c. \end{aligned}$$

Lagrange observes that the greater u is, the nearer will $a\,S_{u-1}$ approach to S_u, and therefore

$$a = \frac{S_u}{S_{u-1}} \ldots . (2)$$

will be an approximation to the root a, so much the nearer as u is greater. But this is not necessarily the case, unless the roots are all positive, for when some of the roots are negative, their powers will appear in S_u and S_{u-1}, with contrary signs, and, therefore, whichever of the numbers u and $u - 1$ is even, the corresponding sum, S_u, or S_{u-1} may exceed the other in any ratio. If, however, we take two consecutive sums in which the exponents are even and find the square root of their quotient, the desired approximation will be obtained; thus we shall have nearly

$$a = \sqrt{\frac{S_u}{S_{u-2}}} \ldots . (3),$$

u being an even number. Let us, for example, suppose the roots of an equation to be 3, — 2, and — 2, then

$$S_2 = 9 + 4 + 4 = 17$$
$$S_3 = 27 - 8 - 8 = 11$$
$$S_4 = 81 + 16 + 16 = 113$$
$$S_5 = 243 - 32 - 32 = 179$$
$$S_6 = 729 + 64 + 64 = 857$$

It is obvious that the formula (2) affords no approximation to the root 3; the formula (3) gives, when $u = 4$,

$$a = \sqrt{\frac{113}{17}} = 2 \cdot 69,$$

and when $u = 6$,

$$a = \sqrt{\frac{857}{113}} = 2 \cdot 75,$$

and so on. The method of Newton is free from error, but is much less convergent than that above. See the *Universal Arithmetic*, or the *Supp. to the Ency. Britt.* art. *Equations*.

NOTE (page 4).

The value of the base a in the Napierian system may be determined by reverting the series

$$x = a - 1 - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.$$

which gives for a the expression (*see Alg.* p. 253)

$$a = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot3} + \frac{x^4}{2\cdot3\cdot4} + \&c.$$

This, when $x = 1$, furnishes for the Napierian base e the value

$$e = 1 + 1 + \tfrac{1}{2} + \frac{1}{2\cdot3}\frac{1}{2\cdot3\cdot4} + \&c.$$

The same expression may be obtained from the *exponential theorem*, which is a formula exhibiting the development of a^x, according to the ascending powers of x. This theorem we shall here establish, first showing that the form of the development proposed is possible. In order to this, put a^x under the form $(1 + a - 1)^x$; then, by the binomial theorem, we have

$$\{1 + (a-1)\}^x = 1 + x(a-1) + \frac{x(x-1)}{1\cdot2}(a-1)^2 + \frac{x(x-1)(x-2)(a-1)^3}{1\cdot2\cdot3} + \&c.$$

The right hand member of this equation may evidently be arranged in the form

$$1 + \{(a-1) - \frac{(a-1)^3}{2} + \frac{(a-1)^3}{3} - \&c.\}\,x + Bx^2 + Cx^3 + Dx^4 + \&c.$$

which is according to the ascending powers of x. Assume, then,

$$a^x = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c. \dots (1)$$

$$\therefore a^y = 1 + Ay + By^2 + Cy^3 + Dy^4 + \&c.$$

$$\therefore a^x - a^y = A(x-y) + B(x^2-y^2) + C(x^3-y^3) + D(x^4-y^4) + \&c. \dots (2),$$

also

$$a^{x-y} = 1 + A(x-y) + B(x-y)^2 + C(x-y)^3 + D(x-y)^4 + \&c.$$

or, transposing the 1, and multiplying each member by a^y,

$$a^x - a^y = a^y\{A(x-y) + B(x-y)^2 + C(x-y)^3 + D(x-y)^4 + \&c.\} \dots (3).$$

Hence the second members of the equations (2) and (3) are equal, therefore, dividing each by $x - y$, and then supposing in the quotients $x = y$, we have, since

$$a^y = a^x = 1 + Ax + Bx^2 + \&c.$$

this equation, viz.

$$A + 2Bx + 3Cx^2 + 4Dx^3 + \&c. = A\,(1 + Ax + Bx^2 + Cx^3 + \&c.)$$

therefore, comparing the coefficients of the like powers of x, we have

$$B = \frac{A^2}{2},\ C = \frac{A^3}{2 \cdot 3},\ D = \frac{A^4}{2 \cdot 3 \cdot 4},\ \&c.$$

Hence from (1)

$$a^x = 1 + Ax + \frac{A^2x^2}{2} + \frac{A^3x^3}{2 \cdot 3} + \frac{A^4x^4}{2 \cdot 3 \cdot 4} + \&c.$$

in which

$$A = a - 1 - \tfrac{1}{2}\,(a - 1)^2 + \tfrac{1}{3}\,(a - 1)^3 - \&c.$$

When $A = 1$, the base becomes the Napierian, and then

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4} + \&c.$$

and for $x = 1$,

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \&c. = 2 \cdot 71828182846.$$

THE END.

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